King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 202 Major Exam I
The First Semester of 2009-2010 (091)

Time Allowed: 90 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		6
2		10
3		10
4		10
5		10
6		10
7		10
Total		66

Q:1 (a) (3 points) Show that $y = c_1 e^x + c_2 e^{-x}$ is a two paremeter family of solutions of

$$y'' - y = 0.$$

Sol. $y = c_1 e^x + c_2 e^{-x}$, $y' = c_1 e^x - c_2 e^{-x}$, and $y'' = c_1 e^x + c_2 e^{-x}$

$$y'' - y = c_1 e^x + c_2 e^{-x} - c_1 e^x - c_2 e^{-x} = 0$$

(b) (3 points) Find a member of this family that satisfy: $y(1) = 0$ and $y'(1) = e$.

Sol. $y(1) = 0 \ \Rightarrow c_1 e + c_2 e^{-1} = 0 \qquad (1)$

$y'(1) = e \ \Rightarrow c_1 e - c_2 e^{-1} = e \qquad (2)$

$(1) + (2) \ \Rightarrow 2c_1 e = e \ \Rightarrow c_1 = \frac{1}{2}$

$(1) - (2) \ \Rightarrow 2c_2 e^{-1} = -e \ \Rightarrow c_2 = -\frac{e^2}{2}$

$y = \frac{1}{2} e^x - \frac{e^2}{2} e^{-x} = \frac{e^x - e^{-x+2}}{2}.$

Q:2 (10 points) Solve the differential equation

$$\frac{dy}{dx} = \frac{2xy + y - 2x - 1}{3xy - y + 3x - 1}$$

Sol. $\frac{dy}{dx} = \frac{(2x+1)(y-1)}{(3x-1)(y+1)} \ \Rightarrow \frac{y+1}{y-1} dy = \frac{2x+1}{3x-1} dx$

$$\left(1 + \frac{2}{y-1}\right) dy = \left(\frac{2}{3} + \frac{\frac{5}{3}}{3x-1}\right) dx$$

$$y + 2\ln|y-1| = \frac{2}{3}x + \frac{5}{9}\ln|3x-1| + C_1$$

$$\ln(y-1)^2 - \ln(3x-1)^{\frac{5}{9}} = \frac{2}{3}x - y + C_1$$

$$\ln\left(\frac{(y-1)^2}{(3x-1)^{\frac{5}{9}}}\right) = \frac{2}{3}x - y + C_1$$

$$\frac{(y-1)^2}{(3x-1)^{\frac{5}{9}}} = e^{\frac{2}{3}x - y + C_1} = Ce^{\frac{2}{3}x - y}$$

Q:3 (10 points) Solve the IVP

$$(x+1)\frac{dy}{dx}+(x+2)\,y=2xe^{-x}, \qquad y(0)=1$$

Sol. $\dfrac{dy}{dx}+\dfrac{x+2}{x+1}y=\dfrac{2xe^{-x}}{x+1}$ $\qquad (*)$

$P(x)=\dfrac{x+2}{x+1}=1+\dfrac{1}{x+1}$ and $IF=e^{\int\left(1+\frac{1}{x+1}\right)dx}=e^{x+\ln(x+1)}=(x+1)\,e^{x}$

Multiplying both sides of $(*)$ by IF we get $\dfrac{d}{dx}\left((x+1)\,e^{x}y\right)=2x$

$(x+1)\,e^{x}y=x^{2}+C$

$y(0)=1\Rightarrow C=1$ and the solution is $y=\dfrac{\left(x^{2}+1\right)e^{-x}}{x+1}$.

Q:4 Given the following differential equation

$$8xy^{2}\;dx+\left(12x^{2}y+40y^{2}-2\right)dy=0.$$

(a) (2 points) Determine if the differential equation is EXACT or not.

Sol. $M(x,y)=8xy^{2}$, $N(x,y)=12x^{2}y+40y^{2}-2$

$M_{y}=16xy$, and $N_{x}=24xy$.

Since $M_{y}\neq N_{x}$, therefore the give DE is not EXACT.

(b) (4 points) Express the given differential equation as an exact equation by multiplying with an appropriate Integrating Factor.

Sol. $\dfrac{N_{x}-M_{y}}{M}=\dfrac{24xy-16xy}{8xy^{2}}=\dfrac{8xy}{8xy^{2}}=\dfrac{1}{y}.$

$\mu(x,y)=e^{\int\frac{1}{y}dy}=e^{\ln y}=y$

Multiplying by $\mu(x,y)$, we get the exact equation

$$8xy^{3}\;dx+\left(12x^{2}y^{2}+40y^{3}-2y\right)dy=0.$$

(c) (4 points) Solve the exact differential equation obtained in (b).

Sol. $\dfrac{\partial f}{\partial x}=M(x,y)=8xy^{3}$ and by integrating we get $f(x,y)=4x^{2}y^{3}+g(y)$

Now $\dfrac{\partial f}{\partial y}=12x^{2}y^{2}+g'(y)=N(x,y)=12x^{2}y^{2}+40y^{3}-2y$

$\Rightarrow g'(y)=40y^{3}-2y\ \Rightarrow g(y)=10y^{4}-y^{2}+C$

The one parameter family of solutions is $4x^{2}y^{2}+10y^{4}-y^{2}+C=0$.

Q:5 (*a*) (5 points) Find a suitable substitution that transforms the differential equation

$$\left(\sin x - y^2 \cos x\right) dx + \frac{1}{y} dy = 0.$$

into a **LINEAR** differential equation. Find the new linear equation **but do not solve it**.

Sol. Given equation can be written as $\dfrac{dy}{dx} + (\sin x)\, y = \cos x \ y^3$ which is a Bernoulli'e equation.with $n = 3$.

Let $u = y^{1-3} = y^{-2}$ or $y = u^{-\frac{1}{2}}$ and $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = -\dfrac{1}{2}u^{-\frac{3}{2}}\dfrac{du}{dx}$.

$$-\frac{1}{2}u^{-\frac{3}{2}}\frac{du}{dx} + (\sin x)\, u^{-\frac{1}{2}} = (\cos x)\, u^{-\frac{3}{2}}$$

$\dfrac{du}{dx} - (2\sin x)\, u = \cos x$ which is a LINEAR equation.

(*b*) (5 points) Find a suitable substitution that transforms the differential equation

$$\left(2x^2 + 3xy\right) dx + 5y^2 dy = 0$$

into a SEPARABLE differential equation. Find the new separable equation **but do not solve it**.

Sol. $M(x, y) = 2x^2 + 3xy$ and $N(x, y) = 5y^2$ are homogeneous functions of degree 2 and therefore the given equation is also homogeneous.

Put $y = ux$ and $dy = udx + xdu$ is the given equation

$$\left(2x^2 + 3x^2 u\right) dx + 5x^2 u^2 (xdu + udx) = 0$$

$$(2 + 3u)\, dx + 5u^2 (xdu + udx) = 0$$

$$\left(5u^3 + 3u + 2\right) dx + 5u^2 x dx = 0$$

$\dfrac{1}{x}dx = -\dfrac{5u^2}{5u^3 + 3u + 2}$ which is a separable equation.

Q6 (10 points) A small metal bar initially at 75^oF is placed in a freezer. The freezer is kept at the constant temperature 35^oF. After one minute the temperature of the metal bar is 55^oF. Find the exact time needed for the temperature of the metal bar to reach 45^oF after it is placed in the freezer.

Sol. $T(0) = 75,\ T_m = 35,\ T(1) = 55$ and we need to find $T(?) = 45$

$$\frac{dT}{dt} = (T - T_m)\,k \ \Rightarrow \frac{1}{T - T_m}dT = kdt$$

$$\ln(T - T_m) = kt + C$$

$$T - T_m = e^{kt+C_1} = Ce^{kt} \ \Rightarrow T = T_m + Ce^{kt} = 35 + Ce^{kt}$$

$$T(0) = 75 \ \Rightarrow 75 = 35 + C \ \Rightarrow C = 40 \text{ and } T = 35 + 40e^{kt}$$

$$T(1) = 55 \ \Rightarrow 55 = 35 + 40e^{k} \ \Rightarrow e^{k} = \frac{20}{40} = \frac{1}{2} \text{ and } e^{kt} = \left(\frac{1}{2}\right)^t$$

$$T = 35 + 40\left(\frac{1}{2}\right)^t$$

$$T(?) = 45 \ \Rightarrow 45 = 35 + 40\left(\frac{1}{2}\right)^t \ \Rightarrow \left(\frac{1}{2}\right)^t = \frac{1}{4}$$

$$t\ \ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{4}\right) \ \Rightarrow t = \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{1}{2}\right)} = \frac{\ln 4}{\ln 2} = \log_2 4 = 2.$$

Q:7 (10 points) Verify that $y_1(x) = 2\sin x + 3\cos x$ and $y_2(x) = \sin x - \cos x$ are solutions of $y'' + y = 0$. Also detremine if $\{y_1(x), y_2(x)\}$ form a fundamental set of solutions on $[0, 2\pi]$.

Sol. $y_1(x) = 2\sin x + 3\cos x$

$$y_1'(x) = 2\cos x - 3\sin x \text{ and } y_1''(x) = -2\sin x - 3\cos x = -y_1(x) \ \Rightarrow y_1'' + y_1 = 0.$$

$$y_2(x) = \sin x - \cos x$$

$$y_2'(x) = \cos x + \sin x \text{ and } y_2''(x) = -\sin x + \cos x = -y_2(x) \ \Rightarrow y_2'' + y_2 = 0.$$

$$W(y_1, y_2) = \begin{vmatrix} 2\sin x + 3\cos x & \sin x - \cos x \\ 2\cos x - 3\sin x & \cos x + \sin x \end{vmatrix}$$

$$= (2\sin x + 3\cos x)(\cos x + \sin x) - (\sin x - \cos x)(2\cos x - 3\sin x)$$

$$= 5\cos^2 x + 5\sin^2 x = 5 \neq 0 \text{ for all } x.$$

Hence $\{y_1(x), y_2(x)\}$ form a fundamental set of solutions.