1. Integration in Banach Spaces

In this section, $(\Omega, \mathcal{A}, \mu)$ is a finite measure space, X is a Banach space over $\Bbbk$ with norm $\|\cdot\|$. We will develop the threory of integration in Banach spaces in parallel to that of classical analysis.

Definition 1. *Let $E \subset \Omega$ be a measurable set. A function $\chi_E : \Omega \to X$ is called a step function if there exists a $b \in X$ such that*

$$\chi_E(t) = \begin{cases} b, & t \in E \\ 0, & t \notin E \end{cases}.$$

A function $u : \Omega \to X$ is called a simple function if there exists a finite sequence of pairwise disjoint measurable sets $E_1, E_2, \cdots, E_n \subset \Omega$ and a sequence of scalers $\alpha_1, \alpha_2, \cdots, \alpha_n \in \Bbbk$ such that

$$u(t) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(t) \ \ \forall t \in \Omega.$$

Definition 2. *A function $u : \Omega \to X$ is called μ-measurable if it is the pointwise limit of a sequence of simple functions. i.e., if there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of simple functions such that $\|u_n(t) - u(t)\| \to 0$ for almost all $t \in \Omega$.*

All functions to be considered from this point on are μ-measurable.

Definition 3. *Let $u : \Omega \to X$ be a simple function.. The integral of u with respect to the measure μ is defined to be*

$$\int_{\Omega} u(t)\, d\mu = \sum_{i=1}^{n} \alpha_i b_i \mu(E_i).$$

Notice that the integral of a simple function is an element of the Banach space X .

Definition 4. *A function $u : \Omega \to X$ is called "Bochner" integrable if there exists a sequence of simple functions $\{u_n\}_{n=1}^{\infty}$ such that $\|u_n(t) - u(t)\| \to 0$ for almost all $t \in \Omega$ and*

$$\lim_{n \to \infty} \int_{\Omega} u_n(t)\, d\mu$$

exists in X. In this case the limit is denoted by

$$\int_{\Omega} u(t)\, d\mu.$$

Proposition 5.

1. *If $u : \Omega \to X$ is integrable, then $\int_\Omega u(t)\, d\mu$ is independent of the choice of the sequence of simple functions converging to u.*

2. *$\int_\Omega u(t)\, d\mu$ exists if and only if $\int_\Omega \|u(t)\|\, d\mu$ exists.*

The following examples are direct generalizations of the classical L^p spaces, $1 \leq p \leq \infty$ and C^k spaces, ... etc.

Example 6. *Let $\Omega = I = [a, b]$ be a finite integral and μ the Lebesgue measure. (we will write dt for $d\mu$)*

1. *The spaces $L^p(I; X)$, $1 \leq p < \infty$ is defined to be*

$$L^p(I; X) = \left\{ u : I \to X : \int_a^b \|u(t)\|^p\, dt < \infty \right\}.$$

This space is a Banach space with the norm

$$\|u\|_{L^p(I;X)} = \left(\int_a^b \|u(t)\|^p\, dt \right)^{1/p}.$$

In particular, if X is a Hilbert space, the space $L^2(I; X)$ is also a Hilbert space with the inner product

$$\langle u, v \rangle_{L^p(I;X)} = \int_a^b \langle u(t), v(t) \rangle\, dt.$$

The dual space $(L^p(I; X))^*$ can be identified with $L^q(I; X^*)$. The space $L^\infty(I; X)$ is defined to be

$$L^\infty(I; X) = \left\{ u : I \to X : \operatorname*{essup}_{t \in I} \{\|u(t)\|\} < \infty \right\}.$$

It is a Banach space under the essential sup norm.

2. *Recall that the Frechet derivative of a function $u : U \subseteq Y \to X$ is a continuous linear operator on Y inro X. In the case Y is the space of real numbers and $U = I$, $u'(t)$ is a linear operator on real numbers into X. This is simply multiplication of an element of X by the real number. Hnece,*

$u'(t)$ can be identified with an element of X. It follows that the dervative can be regarded as a function $u' : I \to X$. In this sense $C^m(I;X)$ is defined to be the set of functions that are continuous together with their first m derivatives from I into X. It becomes a Banach space in the norm

$$\|u\|_{C^m(I;X)} = \sum_{k=0}^{m} \sup_{t\in I} \left\|u^{(k)}(t)\right\|.$$

2. Application to the Taylor Theorem

In this section we show that the classical Taylor Theorem with remainder takes essentially the same form in Banach spaces. In the following theorem X, Y are Banach spaces over $\mathbb{k}$ and $U \subseteq X$ is open and convex. Before stating the theorem we note that if $g : I \to Y$ is continuous and I is compact then we can get a sequence of simple functions g_n which converges uniformly to g. i.e., $\max_{t\in I} \|g_n(t) - g(t)\|_Y \to 0$ as $n \to \infty$. Furthermore, each g_n is supported (i.e., nonzero) on a finite set of pair-wise disjoint subintervals of I. (see the comment after the theorem.

Theorem 7. *Suppose $f : U \subseteq X \to Y$ is C^{n+1} on U. Then, for every $x \in U$ and every $h \in X$ such that $x + h \in X$, the Taylor formula*

$$f(x+h) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) h^k + R_n(x) \tag{1}$$

holds, where

$$\begin{aligned} f^{(k)}(x) h^k &= f^{(k)}(x) hh\cdots h, \\ f^{(0)}(x) h^0 &= f(x) \end{aligned}$$

and

$$R_n(x) = \int_0^1 \frac{(1-\tau)^n}{n!} f^{(n+1)}(x+\tau h) h^{n+1} d\tau. \tag{2}$$

Proof. For $\eta \in Y^*$ set

$$\varphi(t) = \langle \eta, f(x+th) \rangle, \ t \in [0,1].$$

Then

$$\varphi^{(k)}(t) = \left\langle \eta, f^{(k)}(x+th)\,h^k \right\rangle, \; k = 0,1,2,\cdots,n+1, \quad t \in [0,1].$$

Applying the classical Taylor Theorem to the function φ we get

$$\varphi(1) = \sum_{k=0}^{n} \frac{1}{k!}\varphi^{(k)}(0) + \int_0^1 \frac{(1-\tau)^n}{n!}\varphi^{(n+1)}(\tau)\,d\tau.$$

Hence,

$$\langle \eta, f(x+h) \rangle = \left\langle \eta, \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) + \int_0^1 \frac{(1-\tau)^n}{n!} f^{(n+1)}(x+\tau h)\,h^{n+1} d\tau \right\rangle.$$

Since $\eta \in Y^*$ is arbitrary, we get (1), (2). ■

Corollary 8. $\|R_n(x)\| \leq \frac{1}{(n+1)!}\sup_{\tau\in[0,1]} \left\| f^{(n+1)}(x+\tau h)\,h^{n+1} \right\|.$

Some comments are now in order. Notice that in the proof of the Taylor formula (1) above we needed to switch the integration with the pairing with η. To justify this let's define the function $g : [0,1] \to Y$ by $g(t) = f^{(n+1)}(x+th)$ and let g_n be a sequence of simple functions uniformly converging to g. For each n we have

$$\begin{aligned} g_n(t) &= \sum_{i=1}^{n} b_{n,i}\chi_{\triangle_{n,i}} \\ \int_0^1 g_n(t)\,dt &= \sum_{i=1}^{n} b_{n,i} m(\triangle_{n,i}) \end{aligned}$$

then

$$\begin{aligned} \left\langle \eta, \int_0^1 g_n(t)\,dt \right\rangle &= \left\langle \eta, \sum_{i=1}^{n} b_{n,i} m(\triangle_{n,i}) \right\rangle \\ &= \sum_{i=1}^{n} m(\triangle_{n,i}) \langle \eta, b_{n,i} \rangle = \sum_{i=1}^{n} \int_{\triangle_i} \langle \eta, b_{n,i} \rangle\,dt \\ &= \sum_{i=1}^{n} \int_{\triangle_i} \langle \eta, \chi_{\triangle_i}(t) \rangle\,dt = \sum_{i=1}^{n} \int_0^1 \langle \eta, \chi_{\triangle_i}(t) \rangle\,dt \\ &= \int_0^1 \left\langle \eta, \sum_{i=1}^{n} \chi_{\triangle_i}(t) \right\rangle dt = \int_\Omega \langle \eta, g_n(t) \rangle\,dt. \end{aligned}$$

Then

$$\begin{aligned}
\left\langle \eta, \int_{\Omega} g(t)\, dt \right\rangle &= \left\langle \eta, \lim \int_0^1 g_n(t)\, dt \right\rangle \\
&= \lim \left\langle \eta, \int_0^1 g_n(t)\, dt \right\rangle = \lim \int_0^1 \langle \eta, g_n(t) \rangle\, dt \\
&= \int_0^1 \lim \langle \eta, g_n(t) \rangle\, dt = \int_0^1 \langle \eta, g(t) \rangle\, dt.
\end{aligned}$$

In the above string of equations switching the limit with the pairing with η is justified by the continuity of η. Switching the limit with the integration can be seen as follows

$$\begin{aligned}
& \left| \int_0^1 \left(\langle \eta, g_n(t) \rangle - \langle \eta, g(t) \rangle \right) dt \right| \\
\leq & \int_0^1 \left| \langle \eta, g_n(t) \rangle - \langle \eta, g(t) \rangle \right| dt \\
\leq & \|\eta\| \int_0^1 \|g_n(t) - g(t)\|\, dt \\
\leq & \|\eta\| \max_{t \in I} \|g_n(t) - g(t)\| \to 0.
\end{aligned}$$