1 Trace Theory

Here, we consider $1 \leq p < +\infty$ and Ω be an open of $\mathbb{R}^N$.
Lemma. Let $\Omega = \mathbb{R}^N_+$, there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^{N-1}} |u(x^1, 0)|^p dx^1\right)^{\frac{1}{p}} \leq C\|u\|_{w^{1,p}(\mathbb{R}^N)}, \quad \forall u \ \in \ C_0^1(\mathbb{R}^N)$$

Proof. Set $G(t) = |t|^{p-1}t$ and $u \ \in \ C_0^1(\mathbb{R}^N)$. We have

$$\begin{aligned} G(u(x', 0)) &= -\int_0^{+\infty} \frac{\partial}{\partial x_N}\left(G\left(u(x', x_N)\right)\right) dx_N \\ &= -\int_0^{+\infty} G'(u(u', x_N))\frac{\partial u}{\partial x_N}(x', x_N) dx_N \end{aligned}$$

That is,

$$|u(x', 0)|^p \leq p\int_0^{+\infty} |u(x', x_N)|^{p-1}\left|\frac{\partial u}{\partial x_N}(x', x_N)\right| dx_N$$

So, we have by Young's inequality

$$|u(x', 0)|^p \leq C\left[\int_0^{+\infty} |u(x)|^p dx_N + \int_0^{+\infty}\left|\frac{\partial u}{\partial x_N}(x', x_N)\right|^p dx_N\right]$$

By integrating over $\mathbb{R}^{N-1}$, we arrive at

$$\int_{\mathbb{R}^{N-1}} |u(x', 0)|^p dx' \leq C\left[\|u\|_p^p + \left\|\frac{\partial u}{\partial x_N}\right\|_p^p\right].$$

This completes the proof.

We define the operator

$$G : C_0^1(\mathbb{R}^N) \longrightarrow L^p(\Gamma),$$

where $\Gamma : \left\{(x', 0) \ / \ x' \ \in \ \mathbb{R}^{N-1}\right\} = \partial\Omega$.

This operator, which takes $u \longrightarrow u_{|\partial\Omega}$ is linear continuous. Since $C_0^1(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, we then extend it by continuity to $W^{1,p}(\mathbb{R}^N)$.

This operator is called, by definition, the trace operator.

Remark 1. It is clear, from the lemma, that the extension cannot be taken from $L^P(\mathbb{R}^N_+)$ to $L^p(\Gamma)$. This shows that L^P functions do not necessarily have trace on Γ.

Remark 2. In the case Ω is bounded, we use the local coordinates to define the trace of $u \in W^{1,p}(\Omega)$ over $\Gamma = \partial\Omega$.
Theorem. Suppose that Ω is a domain of$\mathbb{R}^N$ of class C^m. Suppose that there exists an extension operator

$$E : W^{m,P}(\Omega) \to W^{m,P}(\mathbb{R}^N).$$

Then

(i) If $mp < n, \quad W^{M,p}(\Omega) \longrightarrow L^q(\partial\Omega), \quad \forall p \leq q \leq \frac{(N-1)p}{N-mp}$.

(ii) If $mp = N, \quad W^{m,P}(\Omega) \longrightarrow (\partial\Omega), \quad \forall p \leq q < +\infty$.

Proof. See Adams p. 114.
Corollary. Suppose that Ω is of class C^1. Then we have

(i) For $p < N$, $W^{1,p}(\Omega) \longrightarrow L^q(\partial\Omega)$, $\forall p \leq q < \frac{(N-1)p}{N-p}$

(ii) For $p = N$, $W^{1,p}(\Omega) \longrightarrow L^q(\partial\Omega)$, $\forall p \leq q < +\infty$.

Remark. The above embeddings are continuous. That is $\exists C > 0$ such that

$$\|u_{|\partial\Omega}\|_{L^q(\partial\Omega)} \leq C\|u\|_{W^{m,p}(\Omega)}.$$

Theorem. Suppose that Ω is a domain of class C^1 in $\mathbb{R}^N$.

(i) If $u \in W^{1,P}(\Omega)$ then $u_{|\Gamma} \in W^{1-\frac{1}{p},p}(\partial\Omega)$, with

$$\|u_{|\Gamma}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

(ii) Conversely, if $v \in W^{1-\frac{1}{p},p}(\partial\Omega)$ then there exists $u \in W^{1,p}(\Omega)$ such that $u_{|\Gamma} = v$ and

$$\|u\|_{W^{1,p}(\Omega)} \leq C^1\|v\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}$$

Remark. This last theorem has a generalization to functions of $W^{m,p}(\Omega)$. In this case, we have to assume the existence of an extension operator. See Adams p.215-217.
Definition. (Sobolev spaces of fractional orders)

Suppose that $s \in (0,1)$ and $1 \leq p < +\infty$, we define

$$W^{s,p}(\Omega) = \left\{ u \,\epsilon\, L^p(\Omega) / \frac{|u(x) - u(y)|}{|x = y|^{s+\frac{N}{p}}} \in L^p(\Omega \times \Omega) \right\}$$

Example. Let

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 / x^2 + y^2 < 1 \right\}.$$

We define

$$u(x, y) = (x^2 + y^2)^\alpha, \ \alpha > 0$$

We easily see that $u \in H^2(\Omega)$, since $\frac{\partial u}{\partial x} = 2\alpha x (x^2 + y^2)^{\alpha - 1}$.

So

$$\int_\Omega \left| \frac{\partial u}{\partial x} \right|^2 = 4\alpha^2 \int_0^{2\pi} \int_0^1 r^2 \cos^2 \theta (r^2)^{2\alpha - 2} r \ dx \ d\theta$$
$$= 4\alpha^2 \int_0^{2\pi} \cos^2 \theta \ d\theta \int_0^1 r^{4\alpha - 1} dr = \alpha\pi, \ \text{a=if } \alpha > 0.$$

Similarly $\int_\Omega \left| \frac{\partial u}{\partial y} \right|^2 = \alpha\pi$.

$$v = u_{|\Gamma} = 1 \in W^{\frac{1}{2},2}(\partial\Omega) = H^{\frac{1}{2}}(\partial\Omega),$$

where $\partial\Omega = \{x^2 + y^2 = 1\}$. It is clear $\int_{\partial\Omega} v^2 = 2\pi$. Also

$$\frac{|u(x) - u(y)|}{|x - y|^{\frac{1}{2} + \frac{2}{2}}} = 0 \in L^2(\Gamma \times \Gamma)$$