1 Maximum Principle

Let $I = (0, 1)$ and consider the problem

$$(P)\begin{cases} -u''(x) + u(x) = f(x), & x \in I \\ u(0) = \alpha, & u)1) = \beta \end{cases}$$

It is well-known that if $f \in L^2(I)$ then P has a unique solution $u \ \Omega \ H^2(I)$, which satisfies

$$\int_0^1 (u'\phi' + u\phi)dx = \int_0^1 f\ dx, \quad \forall \varphi \in H_0^1(I).$$

Theorem: The solution $u \in H^2(I)$ of (P) satisfies

$$min\left\{\alpha, \beta, \inf_I f\right\} \le u(x) \le \max\left\{\alpha, \beta, \sup_I f\right\}, \quad \forall x \in I.$$

Proof: Define a C^1-function G such that

(i) G is strictly increasing on $(0, +\infty)$

(ii) $G(x) = 0$ on $(-\infty, 0]$.

Let

$$k = \max\left\{\alpha, \beta, \sup_I f\right\}$$

If $k = +\infty$ then $u(x) \le k$.
If $k < +\infty$. Then, Let $v(x) = G(u - K)$.

Since $u - k \in H^1$, G is C^1, and $G(0) = 0$ then $v \in H^1(I)$. Moreover,

$$u(0) - \kappa \le 0 \Rightarrow v(0)$$

and

$$u(1) - \kappa \le 0 \Rightarrow v(1) = 0$$

So $v \in H_0^1(I)$; hence, we have

$$\int_0^1 u'v' + uv = \int_0^1 fv$$

$$\int_0^1 u'^2 G'(u - k) + \int_0^1 uG(u - k) = \int_0^1 fG(u - k).$$

This gives

$$\int_0^1 u'^2 G'(u - k) + \int_0^1 (u - k)G(u - k) = \int_0^1 (f - k)G(u - k)$$

But $f - k \le 0$ and $G(u - k) \ge 0$. So,

$$\int_0^1 (u - k)G(u - k) = -\int u'^2 G'(u - k) + \int (f - k)G(u - k) \le 0$$

But

$$T = tG(t) \geq 0, \ \forall t \ \in \ \mathbb{R}$$

(G is nondecreasing). This yields

$$(u-k)G(u-k) = 0, \qquad a.e. \ x \ \in \ I.$$

Consequently

$$(u-k) \leq 0, \qquad a.e. \ x \ \in \ I.$$

The continuously of $u(u \ \in C(\bar{I})) \Rightarrow$

$$u \leq k, \ \ \forall x \ \in \ I.$$

To obtain the other part of the inequality, consider $-u$ to be a solution of

$$(P') \begin{cases} -w'' + w = -f \\ w(0) = -\alpha, \ w(1) = -\beta. \end{cases}$$

Corollary: Suppose that $u \in H^2$ satisfies (P). Then

(i) If $\alpha \geq 0, \ \beta \geq 0$, and $f(x) \geq 0, \ \forall x \ \in \ I$. Then $u(x) \geq 0$ over I.

(ii) If $\alpha = \beta = 0$. Then $\|u\|_{L^\infty(I)} \leq \|f\|_{L^\infty(I)}$.

(iii) If $f = 0$. Then $\|u\|_{L^\infty(I)} \leq \max\{|\alpha|, |\beta|\}$.

Exercise: Show that $u \equiv 0$ is the only solution of

$$\begin{cases} -u'' + u = 0, \ \ \text{on } I \\ u(0) = u(1) = 0 \end{cases}$$

Theorem: Let $f \ \in \ L^2(I)$ and suppose $u \ \in \ H^2(I)$ is the solution of

$$\begin{cases} -u'' + u = f, \ \ x \in I \\ u'(0) = u'(1) = 0 \end{cases}$$

Then

$$\inf_I \ f \leq u(x) \leq \sup_I \ f, \ \ \forall x \ \in \ I$$

Proof: We know that

$$\int u'v' + uv = \int fv, \ \ \forall v \ \in \ H^1(I)$$

We set $k = \sup_I \ f$ and let $v = G(u-k)$. By repeating the steps of the previous theorem, The result is established.

Exercise: If $f \ \in \ C(\bar{I})$ and $u \ \in \ C^2(\bar{I})$ is the solution of (P). Use the equation to show that if x_0 is the maximum point of u on $\bar{I}$. Then

$$u(x_0) \leq k = \max\left\{\alpha_1\beta, \max_{\bar{I}} \ f\right\}.$$

Theorem: If $f \in L^2(\mathbb{R})$ and $u \in H^2(\mathbb{R})$ is the solution of $-u'' + u = f$, over $\mathbb{R}$ Then

$$\inf_{\mathbb{R}} \ f \leq u(x) \sup_{\mathbb{R}} \ f.$$