Theorem. *Let $u \in L^p(\Omega)$, $1 < p \leq +\infty$, the following properties are equivalent:*

(i) $u \in W^{1,p}(\Omega)$

(ii) *There exists a constant $c > 0$ such that*

$$\left| \int_\Omega u \frac{\partial \phi}{\partial x_i} \right| \leq C ||\phi||_{L^{p'}(\Omega)}, \quad \forall \phi \in C_0^\infty(\Omega),$$

$$i = 1, 2, \ldots N. \qquad \frac{1}{p'} + \frac{1}{P} = 1.$$

(iii) *There exists a constant $C > 0$, such that for any open $w \subset\subset \Omega$ and any $h \in \mathbb{R}^N$, with $dis(\omega, \Omega^c) > |h|$,* we have $||\tau_h u - u||_{L^p(\omega)} \leq C|h|$

Remarks:

1. In (ii) and (iii), C can be taken to be equal to $||\nabla_u||_{L^p(\Omega)}$
2. When $p = 1$, (ii) does not imply necessarily (i), since functions satisfying (ii) and (iii) are functions of bounded variations, for which the derivatives, in the distributional sens, may be bounded measures. This class of functions is larger that $W^{1,1}(\Omega)$.
3. The proof of this theorem goes exactly like the one in dimension $N = 1$.
4. If Ω is an open and convex. Then for $u \in W^{1,+\infty}(\Omega)$ we have for almost every $(x, y) \in \Omega$:
$$|u(x) - u(y)| \leq ||\nabla u||_\infty \text{dist}(x, y); \qquad (1)$$
hence u has a continuous representative satisfying (1) for all $(x, y) \in \Omega$.
5. If Ω is connected and $\nabla u = 0$ on Ω, then u is constant in Ω.

Theorem (derivative of a product)

Suppose that $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $1 \leq p \leq +\infty$. Then $u\,v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$\frac{\partial}{\partial x_i}(uv) = u\frac{\partial v}{\partial x_i} + v\frac{\partial u}{\partial x_i}$$

Proof. We only repeat the proof of a similar theorem in the case of $N = 1$ by considering $1 \leq p < +\infty$ first. In this case we have $(u_n), (v_n)$ in $C_0^\infty(\mathbb{R}^N)$ such that

1. $u_n \longrightarrow u$ and $v_n \longrightarrow v$ in $L^p(\Omega)$ and hence a.e. in Ω
2. $\nabla u_n \longrightarrow \nabla u$ and $\nabla v_n \longrightarrow \nabla u$ in $L^p(\omega)$, $\forall w \subset\subset \Omega$
3. $||u_n||_\infty \leq ||u||_\infty$, $\quad ||v_n||_\infty \leq ||v||_\infty$.

So

$$\int_\Omega u_n v_n \frac{\partial \phi}{\partial x_i} = -\int_\Omega \left(u_n \frac{\partial v_n}{\partial x_i} + v_n \frac{\partial u_n}{\partial x_i} \right) \phi, \quad \forall \phi \in C_0^\infty(\Omega), \quad \forall i = 1, 2, \ldots, N.$$

By letting $n \longrightarrow \infty$ and noting that $supp\, \phi \subset\subset \Omega$ we easily see that

$$\int_\Omega u_n v_n \frac{\partial \phi}{\partial x_i} = -\int_\Omega \left(u \frac{\partial v}{\partial x_i} + v \frac{\partial u}{\partial x_i} \right) \phi, \quad \forall \phi \in C_0^\infty(\Omega), \quad \forall i = 1, 2, \ldots, N.$$

hence $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.
For $p = +\infty$, $\quad u, v \in W^{1,p}(\omega), \quad \forall p < \infty, \quad w \subset\subset \Omega$ and hence we repeat the same calculations.
Remark. It is necessary that u and v are in $L^\infty(\Omega)$. So that $u\,v \in L^\infty(\Omega)$, since u, v in $W^{1,p}(\Omega)$ are not necesarily bounded in higher dimension spaces.
Example. Let

$$u = (x, y, z) = \begin{cases} \dfrac{x}{(x^2 + y^2 + z^2)} & , \quad (x, y, z) \neq (0, 0, 0) \\ 0 & , \quad (x, y, z) = (0, 0, 0) \end{cases}$$

$$v = \begin{cases} \dfrac{y}{x^2 + y^2 + z^2} & , \quad (x, y, z) \neq (0, 0, 0) \\ 0 & , \quad (x, y, z) = (0, 0, 0) \end{cases}$$

One can easily verify that $u, v \in W^{1,1}(\Omega)$, where $\Omega = \{(u, y, z) \,/\, x^2 + y^2 + z^2 < 1\}$.
However

$$uv = \begin{cases} \dfrac{xy}{(x^2 + y^2 + z^2)^2} & , \quad (x, y, z) \neq (0, 0, 0) \\ 0 & , \quad (x, y, z) = (0, 0, 0) \end{cases}$$

is not in $L^\infty(\Omega)$. In fact $uv \notin W^{1,1}(\Omega)$.
Theorem. (Derivative of a composition)

Suppose that $G \in C^1(\mathbb{R})$ such that $G(0) = 0$ and $|G'(s)| \leq M \quad \forall s \in \mathbb{R}$. *Let* $u \in W^{1,p}(\Omega)$, *then* $Gou \in W^{1,p}(\Omega)$ *with*

$$\frac{\partial}{\partial x_i}(Gou) = (G'ou)\frac{\partial u}{\partial x_i}, \quad 1 \leq i \leq N, \quad (ii)$$

Proof. Since $|G'(s)| \leq M$ and $G(0) = 0$, $\forall s \in \mathbb{R}$, then $|Gou| \leq M|u|$. this implies that $Gou \in L^p(\Omega)$. and

$$\left| (G'ou)\frac{\partial u}{\partial x_i} \right| \leq M \left| \frac{\partial u}{\partial x_i} \right|, \quad 1 \leq i \leq N$$

with $(G'ou)\frac{\partial u}{\partial x_i} \in L^p(\Omega), \quad \forall i = 1, 2, \ldots, N$.

To verify (ii) in the weak sense, we first take $1 \leq p < \infty$. So we know that there exists a sequence (u_n) in $C_0^\infty(\mathbb{R}^N)$ such that $u_n \longrightarrow u$ in $L^p(\Omega)$ and $\nabla u_n \longrightarrow \nabla u$ in $L^p(\omega), \quad \forall w \subset\subset \Omega$. So for $\phi \in C_0^1(\Omega)$ we have

$$\int_\Omega (Gou_n)\frac{\partial \phi}{\partial x_i} = -\int_\Omega (G'ou_n)\phi \frac{\partial u_n}{\partial x_i}, \quad \forall i = 1, 2, \ldots, N.$$

By taking n to ∞ and using the Dominated Convergence Theorem, we obtain

$$\int_\Omega (Gou)\frac{\partial \phi}{\partial x_i} = -\int_\Omega (G'ou)\frac{\partial u}{\partial x_i}\phi, \quad \forall\, i = 1, 2, \ldots, N.$$

Thus we have $Gou \in W^{1,p}(\Omega)$ and (ii) holds.

For $p = +\infty$, we use the fact that $v \in L^\infty(\Omega)$ then $v \in L^p(\Omega')$, $\quad \forall p < \infty, \quad \forall \Omega' \subset\subset \Omega$. We then repeat the above "usual" analysis.

Remark. If Ω is bounded then $G(0) = 0$ is not necessary. Also if $p = +\infty$, we repeat the same analysis for $\Omega' = \Omega$.

1 The Space $W^{m,p}(\Omega)$

Notation: Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_i \in \mathbb{N}, \forall i = 1, 2, \ldots N$, be a multi-index. We denote by

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}, \text{ where } |\alpha| = \alpha_1 + \ldots + \alpha_N.$$

Definition. We define the Sobolev space, for $m \geq 2$,

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \,\text{such that}\, D^\alpha u \in L^p(\Omega), \quad \forall \alpha : |\alpha| \leq m\}.$$

This is the set of all L^p functions, whose derivatives up to order m are L^p functions.

It is easy to see that

$$W^{m,p}(\Omega) = \left\{u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega), \quad \forall i = 1, 2, \ldots, N\right\}$$

Remark. $D^\alpha u$ is a weak derivative of u; that is

$$\int_\Omega u\, D^\alpha\, \phi = (-1)^{|\alpha|} \int_\Omega \phi D^\alpha u.$$

Proposition. $W^{m,p}(\Omega)$ equipped with the norm

$$||u||_{m,p} = ||u||_p + \sum_{1 \leq |\alpha| \leq m} ||D^\alpha u||_p$$

is a Banach space.

Proposition. $W^{m,p}(\Omega)$ is separable, for $1 \leq p < \infty$, and reflexive, for $1 < p < \infty$.

If we denote by $H^m(\Omega) = W^{m,2}(\Omega)$, then we have

Proposition. $H^m(\Omega)$ equipped with the scalar product

$$\langle u, v \rangle = \int_\Omega uv + \sum_{1 \leq |\alpha| \leq m} \int_\Omega D^\alpha u\, D^\alpha v$$

is a Hilbert space.

Remark. (Adams) For Ω sufficiently regular with a bounded boundary $\partial\Omega$, we have, $\forall\varepsilon > 0$ and $1 \leq |\alpha| \leq m-1$ there exists $C > 0$ such that

$$||D^\alpha u||_p \leq \varepsilon \sum_{|\alpha|=m} ||D^\alpha u||_p + C||u||_p, \quad \forall u \in W^{m,p}(\Omega)$$

Consequently, we have in this case,

$$||u||_{m,p} = ||u||_p + \sum_{1\alpha|=m} ||D^\alpha u||_p$$

is an equivalent norm for $W^{m,p}(\Omega)$.

2 Extension Operator

Suppose that $u \in W^{1,p}(\Omega)$. Sometimes it is more convenient to establish some properties by extending u to $\mathbb{R}^N$ by a $W^{1,p}(\mathbb{R}^N)$ function. This is, unfortunately, not always possible. However if Ω is regular this is may be possible.
Notations: Let $x = (x_1, x_2, \ldots, x_{N-1}, x_N) \in \mathbb{R}^N$. We write

$$x = (x', x_N) \text{ with } x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$$

We put

$$|x'| = \left(\sum_{i=1}^{N-1} x_i^2\right)^{\frac{1}{2}}$$

and denote by

$$\begin{aligned}
\mathbb{R}^N_+ &= \{x = (x', x_N) : x_N > 0\}, \text{ the upper hyperplane} \\
Q &= \{x = (x', x_N) : |x'| < 1 \text{ and } |x_N| < 1\}, \text{ A square or cylinder} \\
Q_+ &= Q \cap \mathbb{R}^N_+ \\
Q_0 &= \{x = (x', x_N) : |x'| < 1 \text{ and } x_N = 0\} \text{ unit disk}
\end{aligned}$$

Definition. An open subset $\Omega \subset \mathbb{R}^N$ is said to be of class C^1 if for each $x \in \Gamma = \partial\Omega$, there exists a neighbourhood U of x in $\mathbb{R}^N$ and a bijection $H : Q \longrightarrow U$ such that

$$H \in C^1(\bar{Q}), \quad H^{-1} \in C^1(\bar{U}), \quad H(Q_+) = U \cap \Omega, \text{ and } H(Q_0) = U \cap \Gamma.$$

Notations. Let $f : Q_+ \longrightarrow \mathbb{R}$, we denote by f^* the extension by reflexion of f on Q

$$f^*(x', x_N) = \begin{cases} f(x', x_N), & x_N > 0 \\ f(x', -x_N), & x_N < 0 \end{cases}$$

and

$$f^\#(x', x_N) = \begin{cases} f(x', x_N), & x_N > 0 \\ -f(x', -x_N), & x_N < 0 \end{cases}$$

Lemma. *Let $u \in W^{1,p}(Q_+)$. Then the extension u^* is in $W^{1,p}(Q)$ with*

$$||u^*||_p \leq 2||u||_p, \quad ||u^*||_{W^{1,p}(Q)} \leq 2||u||_{W^{1,p}(Q_+)}$$

Proof. We have to verify that

$$\frac{\partial u^*}{\partial x_i} = \left(\frac{\partial u}{\partial x_i}\right)^*, \quad \forall\, i = 1, 2, \ldots, N-1$$

$$\frac{\partial u^*}{\partial x_N} = \left(\frac{\partial u}{\partial x_N}\right)^\#$$

Let η be a $C^\infty(\mathbb{R})$ function such that

$$\eta(t) = \begin{cases} 0, & t < \frac{1}{2} \\ 1, & t > 1 \end{cases}$$

Define the sequence $\eta_k(t) = \eta(kt), \quad k = 1, 2, 3, \ldots$.Let $\phi \in C_0^1(Q)$; so for $i = 1, 2, \ldots, N-1$ we have

$$\int_Q u^* \frac{\partial \phi}{\partial x_i} = \int_{Q_+} u \frac{\partial \psi}{\partial x_i}, \qquad (iii)$$

where

$$\psi\left(x', x_N\right) = \phi\left(x', x_N\right) + \phi\left(x', -x_N\right)$$

ψ is not necessarily in $C_0^1(Q_+)$ but $\eta_k(x_N)\psi(x', x_N)$ is in $C_0^1(Q_+)$ and

$$\frac{\partial}{\partial x_i}\left(\eta_k\, \psi\right) = \eta_k \frac{\partial \psi}{\partial x_i}, \quad \forall i = 1, 2, \ldots, N-1.$$

Hence

$$\int_{Q_+} \eta_k\, u \frac{\partial \psi}{\partial x_i} = -\int_{Q_+} \frac{\partial u}{\partial x_i} \eta_k \psi, \quad \forall i = 1, 2, \ldots, N-1$$

By using the dominated convergence theorem we get, as $k \longrightarrow \infty$,

$$\int_{Q_+} u \frac{\partial \psi}{\partial x_i} = -\int_{Q_+} \frac{\partial u}{\partial x_i} \psi \qquad (iv)$$

By combining (iii) and (iv) we arrive at

$$\int_Q u^* \frac{\partial \phi}{\partial x_i} = -\int_Q \left(\frac{\partial u}{\partial x_i}\right)^* \phi.$$

Therefore

$$\left(\frac{\partial u}{\partial x_i}\right)^*, \; 1 \leq i \leq N-1$$

are the derivatives of u^*. Also, for $\phi \in C_0^1(Q)$, we have

$$\int_Q u^* \frac{\partial \phi}{\partial x_N} = \int_{Q_+} u \frac{\partial \chi}{\partial x_N},$$

where

$$\chi(x', x_N) = \phi(x', x_N) - \phi(x', -x_N)$$

It is clear that $\chi(x', 0) = 0$, so there exists $M > 0$ such that

$$|\chi(x', x_N)| \le M|x_N|, \quad \forall (x', x_N) \in Q.$$

Since $\eta_k\, \chi \in C_0^1(Q_+)$, we have

$$\int_{Q_+} u \, \frac{\partial}{\partial x_N}\,(\eta_k \chi) = -\int_{Q_+} \frac{\partial u}{\partial x_N}\, \eta_k\, \chi$$

but

$$\frac{\partial}{\partial x_N}\,(\eta_k\, \chi) = \eta_k\, \frac{\partial \chi}{\partial x_N} + k\eta'(kx_N)\chi$$

$$\begin{aligned} \left| \int_{Q_+} uk\eta'(kx_N)\chi\, dx \right| &\le MCk \left| \int_{0<x_N<\frac{1}{k}} x_N u\, dx \right| \\ &\le MC \int_{0<x_N<\frac{1}{k}} |u| dx \longrightarrow 0 \text{ as } k \longrightarrow \infty \end{aligned}$$

Hence

$$\int_{Q_+} u \frac{\partial \chi}{\partial x_N} = -\int_{Q_+} \frac{\partial u}{\partial x_N}\, \chi$$

By noting that

$$\int_{Q_+} \frac{\partial u}{\partial x_N}\, \chi = \int_Q \left(\frac{\partial u}{\partial x_N} \right)^{\#} \phi$$

we arrive at

$$\int_Q u^* \frac{\partial \phi}{\partial x_N} = -\int_Q \left(\frac{\partial u}{\partial x_N} \right)^{\#} \phi$$

Hence $\left(\frac{\partial u}{\partial x_N}\right)^{\#}$ is the weak derivative of u^* with respect to x_N.

Finally it is easy to verify that

$$\begin{aligned} ||u^*||_{L^p(Q)} &\le 2||u||_{L^p(Q_+)} \\ ||u^*||_{W^{1,p}(Q)} &\le 2||u||_{W^{1,p}(Q_+)} \end{aligned}$$

Remark. The above lemma holds if Q_+ is replaced by $\mathbb{R}^N_+$; with no change in the proof.