1 Neumann Problem

Consider the homogeneous Neumann-condition problem

$$\begin{cases} -u'' + u = f, & 0 < x < 1 \\ u'(0) = u'(1) = 0. \end{cases} \tag{1}$$

If u is a classical solution of (1.1), then for each $v \in H^1(I)$, we have

$$\int_0^1 u'v' + uv = \int_0^1 fv. \tag{2}$$

Again by Lax-Millgram lemma, we have a solution u in $H^1(I)$.

If $f \in L^2(I)$, then u is in $H^2(I)$ since, for each $\varphi \in C_0^1(I)$, we have

$$\int_0^1 u'\varphi' = -\int_0^1 (f-u)\varphi,$$

then $u' \in H^1(I)$ with $u'' = f - u$. From (1.2), we obtain

$$\int_0^1 (-u'' + u - f)v + u'(1)v(1) - u'(0)v(0) = 0, \quad \forall\ v \in H^1. \tag{3}$$

First, for $v \in H_0^1(I)$, we have

$$\int_0^1 (-u'' + u - f)v = 0.$$

So $-u'' + u - f = 0$ in $L^2(I)$, hence for almost each $x \in I$. Thus (1.3) is reduced to

$$u'(1)v(1) - u'(0)v(0) = 0, \quad \forall\ v \in H^1(I).$$

Since v is arbitrary, then $u'(1) = u'(0) = 0$.

2 Approximation

Let us consider for $f \in L^2(I)$,

$$\begin{cases} -u'' + u = f, & 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$

We know that (P) has a solution $u \in V = H_0^1(I)$. V is separable, so there exists a basis

$$\{\varphi\}_{k=1}^{\infty} \subset V.$$

We would like to approximate the solution of (P) by solutions in finite-dimension spaces V_n spanned by $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. We thus consider the approximating problem: find $u_n \in V_n$, which satisfies

$$\int_0^1 u_n'v_n' + u_nv_n = \int_0^1 fv_n, \qquad \forall\ v_n \in V_n.$$

This is equivalent to

$$\int_0^1 u_n' \varphi_k' + u_n \varphi_k = \int_0^1 f \varphi_k, \qquad \forall\ k = 1, 2, \ldots, n.$$

Since

$$u_n = \sum_{i=1}^n a_i \varphi_i,$$

then we have

$$\int_0^1 \sum_{i=1}^n a_i \varphi_i' \varphi_k' + \int_0^1 \sum_{i=1}^n a_i \varphi_i \varphi_k = \int_0^1 f \varphi_k, \qquad k = 1, 2, \ldots, n.$$

That is,

$$\sum_{i=1}^n a_i \int_0^1 (\varphi_i' \varphi_k' + \varphi_i \varphi_k) = \int_0^1 f \varphi_k, \quad k = 1, 2, \ldots, n.$$

This is reduced to solving a linear system $MA = f$, where the entries of M are given by $m_{ik} = \int_0^1 (\varphi_i' \varphi_k' + \varphi_i \varphi_k)$ and

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \qquad \text{for } f_k = \int_0^1 f \varphi_k.$$

Remark: Note if the the basis is chosen orthogonal in $H_0^1(I)$ equipped with the inner product

$$\langle u, v \rangle = \int_0^1 (u'v' + uv),$$

then M reduces to the identity I.

Properties of the matrix M:

1) M is symmetric since $m_{ik} = m_{ki}$.

2) M is positive definite:

$$\begin{aligned} a(u_n, u_n) &= \int_0^1 \left[\left(\sum_{i=1}^n a_i \varphi_i' \right) \left(\sum_{k=1}^n a_k \varphi_k' \right) + \left(\sum_{i=1}^n a_i \varphi_i \right) \left(\sum_{i=1}^n a_i \varphi_i \right) \right] \\ &= \sum_{i,k=1}^n a_i a_k \int_0^1 (\varphi_i' \varphi_k' + \varphi_i \varphi_k) = \langle MA, A \rangle. \end{aligned}$$

But a is coercive; so

$$\begin{aligned} a(u_n, u_n) &\geq \|u_n\|^2 = \sum_{i=1}^n a_i^2 \int_0^1 \varphi_i^{'2} + \varphi_i^2 \\ &\geq \alpha \|A\|^2, \quad \alpha = \min \int_0^1 \varphi_i^{'2} + \varphi_i^2. \end{aligned}$$

3 Error Estimation

The variational or weak problem is to find $u \in V$ such that

$$a(u, v) = F(v), \quad \forall\, v \in V,$$

where V is separable Hilbert space, a is a bounded coercive bilinear form, and F is a bounded linear form on V.

The approximating problem is to find $u_h \in V_h$, a finite-dimension subspace of V, such that

$$a(u_h, v_h) = F(v_h), \quad \forall\, v_h \in V_h, \tag{5}$$

Since $v_h \in V_h \subset V$, we also have

$$a(u, v_h) = F(v_h), \quad \forall\, v_h \in V_h. \tag{6}$$

Subtracting (1) from (2), we obtain

$$a(u - u_h, v_h) = 0, \qquad \forall\, v_h \in V_h.$$

Also, $v_h - u_h \in V_h$, so $a(u - u_h, v_n - u_h) = 0$.

Now, we use the coercivity of a to get

$$\begin{aligned}
\alpha \|u - u_h\|_V^2 &\leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h + v_h - u_h) \\
&\leq a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\
&\leq a(u - u_h, u - v_h) \leq M\|u - u_h\|\ \|u - v_h\|_V \\
&\Rightarrow \|u - u_h\|_V \leq \frac{M}{|al}\|u - v_h\|_V, \quad \forall\, v_h \in V_h.
\end{aligned}$$

Remark: In the above example, $M = \alpha = 1$. Therefore, u_h is the projection of u on V_h.

4 Finite elements

We divide $I = (0, 1)$ into regular subintervals with length $h = 1/n$. So,

$$x_0 = 0, \quad x_1 = \frac{1}{n}, \quad x_2 = \frac{2}{n}, \ldots, \quad x_{n-1} = \frac{n-1}{n}, \quad x_n = 1.$$

We define the linear functions or hat functions by

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h}, & x_{i-1} \leq x < x_i \\ \dfrac{x_{i+} - x}{h}, & x_i \leq x \leq x_{i+1} \\ 0, & \text{otherwise} \end{cases}, i = 1, 2, \ldots, n-1.$$

φ_i is a $H_0^1(I)$ function with

$$\varphi_i'(x) = \begin{cases} \dfrac{1}{h}, & x_{i-1} < x < x_i \\ -\dfrac{1}{h}, & x_i < x < x_{i+1} \\ 0, & \text{otherwise .} \end{cases} .$$

Let us compute the matrix entries

$$\begin{aligned}
m_{ii} &= \int_0^1 \varphi_i^2 + \varphi_i'^2 = \int_{x_{i-1}}^{x_{i+1}} (\varphi_i^2 + \varphi_i'^2) \\
&= \int_{x_{i-1}}^{x_i} \frac{1}{h^2}(x - x_{i-11})^2 + \frac{1}{h^2}\int_{x_i}^{x_{i+1}} (x_{i+1} - x)^2 + \frac{1}{h^2}\int_{x_{i-1}}^{x_{i+1}} dx \\
&= \frac{3}{2}h + \frac{2}{h} \\
m_{i,i+1} &= \int_{x_i}^{x_{i+1}} \varphi_i\varphi_{i+1} + \varphi_i\varphi_{i+1}' = \int_{x_i}^{x_{i+1}} \frac{(x_{i+1} - x)(x - x_i)}{h^2} - \frac{1}{h^2}\int_{x_i}^{x_{i+1}} dx \\
&= \frac{h^3}{6}\cdot\frac{1}{h^2} - \frac{1}{h} = \frac{h}{6} - \frac{1}{h}.
\end{aligned}$$

Note that $m_{i+1,i} = m_{i,i+1}$ and $m_{ik} = 0$, for $|i-k| \geq 2$ since $\varphi_i\varphi_k = 0$ and $\varphi_i'\varphi_k' = 0$ on I. Thus

$$M = \frac{1}{h}\begin{bmatrix} 2 + \frac{3}{2}h^2 & -1 + \frac{h}{6} & 0 & \cdots & 0 \\ -1 + \frac{h}{6} & 2 + \frac{3}{2}h^2 & -1 + \frac{h}{6} & \cdots & 0 \\ 0 & & & & \\ \vdots & & & & \\ & & & & -1 + \frac{h}{6} \\ 0 & & 0 & -1 + \frac{h}{6} & 2 + \frac{3}{2}h^2 \end{bmatrix}$$

In our case, we know that the approximate solution and the solution satisfy

$$\|u - u_h\|_V \leq \|u - v_h\|_V, \qquad \forall\, v \in V_h.$$

Let us choose a particular element w_h in V_h such that

$$w_h(x_i) = u(x_i), \quad \forall\, i = 0, 1, 2, \ldots, n.$$

This interpolates u on $\{x_0, x_1, \ldots, x_n\}$. Set

$$e(x) = u(x) - w_h(x);$$

so $e(x_{i+1}) = 0$. Also e is differentiable on (x_i, x_{i+1}), so there exists $\xi_i \in (x_i, x_{i+1})$ such that $e'(\xi_i) = 0$ (Mean Value Theorem). Suppose that u'' exists, then

$$e''(x) = u''(x), \quad x_i < x < x_{i+1}$$

hence $e'(x) = \int_{\xi_i}^x u''(t)dt \Rightarrow$

$$\begin{aligned} |e'(x)| &\leq \int_{\xi_i}^x |u''(t)|dt \leq \left(\left|\int_{\xi_i}^x dt\right|\right)^{1/2} \left(\int_{x_i}^{x_{i+1}} |u''(t)|^2 dt\right)^{1/2} \\ &\leq (x - x_i)^{1/2} \cdot \|u''\|_2, \qquad \forall\, x \in (x_i, x_{i+1}) \end{aligned}$$

Also $e(x) = \int_{x_i}^x e'(t)dt \Rightarrow$

$$\begin{aligned} |e(x)| &\leq \int_{x_i}^{x_{i+1}} |e'(t)|dt \leq \left(\int_{x_i}^{x_{i+1}} dt\right)^{1/2} \left(\int_{x_i}^{x_{i+1}} |e'(x)|^2 dx\right)^{1/2} \\ &\leq h^{1/2} \int_{x_i}^{x_{i+1}} \left(\int_{x_i}^{x_{i+1}} (x - x_i)\|u''\|_2^2\right)^{1/2} \leq h^{1/2}\|u''\|_2 \frac{h}{\sqrt{2}}, \qquad \forall\, x \in (x_i, x_{i+1}). \end{aligned}$$

So

$$|u(x) - w_h(x)| \leq Ch^{3/2}, \quad \forall\, x \in I$$

hence

$$\|u - w_h\|_2 \leq Ch^{3/2}\|u''\|_2$$

By using $|e'(x)| \leq h^{1/2}\|u\|_2$, $\forall\, x \in I$, we obtain

$$\int_0^1 |e'(x)|^2 dx \leq h\|u''\|_2^2$$

Hence

$$\|u - w_h\|_2 \leq h^{1/2}\|u''\|_2.$$

We conclude that

$$\|u - w_h\|_{H^1} \leq h^{1/2}[1 + Ch^2]\|u''\|_2.$$

Hence

$$\|u - w_h\|_{H^1} \leq Ch^{1/2}\|u''\|_2$$

We know that if $f \in L^2(I)$, then $u'' \in L^2(I)$. Consequently, we have
Proposition: If $f \in L^2(I)$, then the solution u and the approximate solution u_h satisfy

$$\|u - u_h\|_{H^1} \leq Ch^{1/2}.$$