1 Dual Space of $W_0^{1,p}(I)$

We denote by $W^{-1,p'}(I)$ the dual space of $W_0^{1,p}(I)$, $1 \leq p < \infty$ and by $H^{-1}(I)$ the dual of $H_0^1(I)$.
Remark: By identifying $L^2(I)$ with its dual, we obtain

$$H_0^1(I) \subset L^2(I) \subset H^{-1}(I)$$

with continuous and dense embedding.
Theorem: Let F be in $W^{-1,p'}(I)$. Then there exist f_0, f_1 in $L^{p'}(I)$ such that

$$\langle F, v\rangle = \int f_0 v + \int f_1 v', \qquad \forall\ v \in W_0^{1,p}(I).$$

Moreover, if I is bounded, f_0 can be taken zero.
Proof: Define the Banach space $E = L^p \times L^p$ equipped with the norm

$$\|h\|_E = \|h_0\|_p + \|h_1\|_p, \quad h = (h_0, h_1) \in E.$$

$T : W_0^{1,p}(I) \to E$ given by $T(u) = (u, u')$ is an isometry.

Let $G = T(W_0^{1,p}(I))$ equipped with the norm of E. We define the linear form $\gamma : G \to \mathbb{R}$ by

$$\gamma(h) = \langle F, T^{-1}h\rangle.$$

It is easy to see that γ is continuous. So, it can be extended to E by Hahn-Banach theorem. Call Φ the extension; hence $\|\Phi\| = \|\gamma\| = \|F\|$. So by Riesz Representation Theorem, there exists $f_0,\ f_1 \in L^{p'}(I)$, such that

$$\Phi(h_0, h_1) = \int f_0 h_0 + f_1 h_1, \quad \forall (h_0, h_1) \in E.$$

In particular, if $u \in W_0^{1,p}(I)$, then

$$\begin{aligned} F(u) &= \langle F, T^{-1}(u, u')\rangle \\ &= \Phi(u, u') = \int f_0 u + f_1 u'. \end{aligned}$$

When I is bounded, $W_0^{1,p}(I)$ is equipped with the norm $\|u\|_{w^{1,p}} = \|u'\|_p$. We then repeat a similar reasoning with $T : W_0^{1,p}(I) \to L^p(I)$ given by $T(u) = u'$.
Remark: f_0 and f_1 are not unique.
Remark: If $v \in C_0^\infty(I)$, then

$$\langle F, v\rangle = \int f_0 v + f_1 v' = \int f_0 v - f_1' v = \int (f_0 - f_1')v.$$

Therefore

$$F = f_0 - f_1' \text{ in } \mathcal{D}'(I).$$

Exercise: Verify that Φ given in the above proof satisfies

$$\|\Phi\|_{E'} = \max\{\|f_0\|_{p'}, \|f_1\|_{p'}\}.$$

1.1 Bilinear forms and Lax-Milgram Lemma

Definition: Let $B : H \times H \to \mathbb{R}$ be a bilinear form on a Hilbert space H. We say that

1) B is continuous if there exists $M > 0$ such that

$$|B(u,v)| \leq M\|u\| \, \|v\|.$$

2) B is coercive (or elliptic) if there exists $\alpha > 0$ such that

$$B(u,u) \geq \alpha\|u\|^2, \qquad \forall \; u \in H.$$

Theorem: (Lax-Milgram Lemma). Given a Hilbert space H, let $B : H \times H \to \mathbb{R}$ be a continuous and coercive bilinear form and $g : h \to \mathbb{R}$ be a continuous (bdd) linear form. Then there exists a unique u in H such that

$$g(v) = B(u,v), \qquad \forall \; v \in H.$$

Application: Consider the problem

$$\begin{cases} -u'' + u = f \text{ in } I = (0,1) \\ \quad u(0) = u(1) = 0. \end{cases} \tag{1.1}$$

For f smooth enough (continuous). This problem can be solved by standard calculus methods. In this case the solution is of class $C^2(I)$. It is called a classical solution. Suppose that f is not regular; say $f \in L^2(I)$ or $f \in H^{-1}(I) =$ dual of $H_0^1(I)$. Is there any solution for (1.1)?

Let $\varphi \in C_0^1(I)$. Multiply φ by equation (1.1) and integrate over I, assuming u to be regular,

$$\int_0^1 u'\varphi' + u\varphi = \int_0^1 f\varphi. \tag{1.2}$$

Question; Is it possible to find u such that (1.2) is satisfied for all $\varphi \in C_0^1(I)$?
Answer: Define the bilinear form $B : H_0^1(I) \times H_0^1(I) \to \mathbb{R}$ by

$$B(u,v) = \int_0^1 u'v' + uv.$$

It is easy to verify that B is continuous and coercive. If $f \in H^{-1}(I)$, we then define the linear form

$$F : H_0^1(I) \to \text{ by } F(v) = \langle f, v \rangle.$$

This is continuous such that $\|F\| = \|f\|_{-1}$. Lax-Milgram lemma guarantees the existence of a unique $u \in H_0^1(I)$ such that

$$B(u,v) = F(v), \quad \forall \; v \in H_0^1(I).$$

That is,

$$\int_0^1 u'v' + uv = \langle f, v \rangle \left(\text{ or } \int_0^1 fv, \text{ if } f \in L^2(I) \right), \quad \forall \; v \in H_0^1(I).$$

Definition: We call the weak formulation of (1): find u in $H_0^1(I)$:

$$\int_0^1 u'v' + uv = \langle f, v\rangle_{H^{-1}\times H_0^1(I)}, \quad \forall\, v \in H_0^1(I). \tag{1.3}$$

Definition: We call $u \in H_0^1(I)$ satisfying (1.3), the weak solution of (1).
Remark: Since $u \in H_0^1(I)$, therefore u is in $C(\overline{I})$; hence $u(0) = u(1) = 0$.
Proposition: If $f \in L^2(I)$, then $u'' = u - f \in L^2(I)$. Thus $u \in H^2(I) \cap H_0^1(I)$. So, we have more regularity that is $u \in C^1(\bar{I})$.
Proof: $\varphi \in C_0^1(I) \Rightarrow$

$$\int_0^1 u'\varphi' = -\int_0^1 (u-f)\varphi,$$

so by definition, u' has a weak derivative $u - f \in L^2(I) \Rightarrow u' \in H^1(I)$, with $u'' = u - f \in L^2(I)$. $\Rightarrow$

$$u \in H_0^1(I) \cap H^2(I).$$

The embedding theorem gives $u' \in C(\overline{I})$; hence $u \in C^1(\overline{I})$.
Exercise: 1) Show that

$$\begin{aligned} &-u'' = \delta \text{ in } I = (-1,1), \\ &u(-1) = u(1) = 0 \end{aligned}$$

has a solution.

2) Solve

$$\begin{aligned} &-u'' + u = f \text{ in } I = (0,1) \\ &u(0) = \alpha \quad u(1) = \beta. \end{aligned}$$

for $f \in L^2(I)$; $\quad \alpha, \beta \in \mathbb{R}$.

1.2 Neumann Problem

Consider the homogeneous Neumann-condition problem

$$\begin{cases} -u'' + u = f, & 0 < x < 1 \\ u'(0) = u'(1) = 0. \end{cases} \tag{1.4}$$

If u is a classical solution of (1.4), then for each $v \in H^1(I)$, we have

$$\int_0^1 u'v' + uv = \int_0^1 fv. \tag{1.5}$$

Again by Lax-Millgram lemma, we have a solution u in $H^1(I)$.

If $f \in L^2(I)$, then u is in $H^2(I)$ since, for each $\varphi \in C_0^1(I)$, we have

$$\int_0^1 u'\varphi' = -\int_0^1 (f-u)\varphi,$$

then $u' \in H^1(I)$ with $u'' = f - u$. From (1.5), we obtain

$$\int_0^1 (-u'' + u - f)v + u'(1)v(1) - u'(0)v(0) = 0, \quad \forall\, v \in H^1. \tag{1.6}$$

First, for $v \in H_0^1(I)$, we have

$$\int_0^1 (-u'' + u - f)v = 0.$$

So $-u'' + u - f = 0$ in $L^2(I)$, hence for almost each $x \in I$. Thus (1.6) is reduced to

$$u'(1)v(1) - u'(0)v(0) = 0, \quad \forall\, v \in H^1(I).$$

Since v is arbitrary, then $u'(1) = u'(0) = 0$.