1 Sobolev Spaces for Higher Orders

Definition: Given an integer $m \geq 2$, we define the Sobolev space of order m by

$$\begin{aligned} W^{m,p}(I) &= \{u \in L^p(I) \quad / \quad u', u", ..., u^{(m)} \in L^p(I)\} \\ &= \{u \in L^p(I) \quad / \quad u' \in W^{m-1,p}(I)\} \end{aligned}$$

The derivatives are in the weak sense. We then set $H^m(I) = W^{m,2}(I)$.
Properties: 1) It is easy to verify that $u \in W^{1,p}(I)$ if and only if there exists m functions $g_1, g_2, \ldots, g_m \in L^p(I)$ such that

$$\int_I u\varphi^{(j)}(x)dx = (-1)^j \int_I g_j \varphi dx, \quad \forall\ \varphi \in C_0^\infty(I).$$

2) The space $W^{1,p}(I)$ equipped with the norm:

$$\|u\|_{m,p} = \|u\|_p + \sum_{j=1}^{m} \|u^{(j)}\|_p$$

is a Banach space.
3) The space $H^m(I)$ equipped with the inner product

$$< u, v >_{m,p} = \int_I (uv + u'v' + \cdots + u^{(m)}v^{(m)})(x)dx$$

is a Hilbert space.
Theorem: $W^{m,p}(I)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.
Theorem: $W^{m,p}(I)$ is continuously embedded in $C^{m-1}(\bar{I})$.

2 The Space $W_0^{1,p}(I)$

Definition: For $1 \leq p < +\infty$, we define $W_0^{1,p}(I)$ to be the closure of $C_0^1(I)$ with respect to the norm of $W^{1,p}(I)$. We denote by $H_0^1(I) = W_0^{1,2}(I)$.
Properties:
1) It is clear that $W_0^{1,p}(I)$ is a Banach space if equipped with the norm of $W^{1,p}(I)$.
2) $H_0^1(I)$ is a Hilbert space with respect to the inner product

$$< u, v >= \int_I (uv + ou'v')(x)dx.$$

3) $W_0^{1,p}(I)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < +\infty$.
4) $W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R})$ since $C_0^1(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, $\ 1 \leq p < \infty$.
Remark: $C_0^1(I)$ (or $C_0^\infty(I)$) is not dense in $L^\infty(I)$; otherwise all L^∞ functions are continuous.
Theorem: Suppose that $u \in W^{1,p}(I)$. Then $u \in W_0^{1,p}(I)$ if and only if $u = 0$ on ∂I (boundary of I).
Proof: 1) Given u in $W_0^{1,p}(I)$, so u is in $C(\bar{I})$. We know there exists a sequence (u_n)

in $C_0^1(I)$ such that $u_n \to u$ in $W^{1,p}(I)$, hence in L^∞ norm. So $\forall\, \varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$|u_n(x) - u(x)| < \varepsilon, \quad \forall\, x \in \bar{I},$$

in particular for $x \in \partial I$. Thus we have $|u(x)| < \varepsilon$ since $u_n(x) = 0, \quad x \in \partial I$. Since ε is arbitrary then $u = 0$ on ∂I.

2) Let G be a C^1 function on $\mathbb{R}$ such that

$$G(s) = \begin{cases} 0, & |s| \leq 1 \\ s, & |s| \geq 2 \end{cases}$$

Given u in $W^{1,p}(I)$, such that $u_{|\partial I} = 0$, we define $u_n = G(nu)/n$. It is easy to verify that $|u_n| \leq |u|$ and $|u_n'| \leq |u'|$; hence $u_n \in W^{1,p}(I)$. Also supp $u_n \subset \{x \in I \;/\; |u(x)| \geq 1/n\}$, which is a compact set of I since $\lim_{x \to \partial I} u(x) = 0$. Thus $u_n \in W^{1,p}(I) \cap C_0(I) \Longrightarrow u_n \in W_0^{1,p}(I)$.

Next, we prove that $u_n \to u$ in $W^{1,p}(I)$. For this we use the dominated convergence theorem.

a) First, note that

$$|u_n - u| \leq 2|u| \qquad |u_n' - u'| \leq (k+1)|u'|.$$

b) It is easy to verify that

$$|u_n - u| \to 0, \qquad |u_n' - u'| \to 0$$

for almost every $x \in I$ and since $2|u| \in L^p$ and $(k+1)|u'| \in L^p(I)$. Then

$$|u_n - u| \to 0 \qquad |u_n' - u| \to 0 \text{ in } L^p(I);$$

hence $u_n \to u$ in $W^{1,p}(I)$. Since $u_n \in W_0^{1,p}(I)$, which is a closed subspace of $W^{1,p}(I)$, then $u \in W_0^{1,p}(I)$.

Theorem:

1) Let $1 < p < \infty$ and $u \in L^p(I)$. Then $u \in W_0^{1,p}(I)$ if and only if there exists a constant C such that

$$|\int_I u\varphi'| \leq C\|\varphi\|_{L^{p'}(I)}, \quad \forall \in \varphi \in C_0^1(I), \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

2) Let $1 \leq p < \infty$ and $u \in L^p(I)$. We define $\tilde{u}$ by

$$\tilde{u}\,(x) = \begin{cases} u(x), & x \in I \\ 0, & x \in \mathbb{R} \backslash I \end{cases}$$

Then $u \in W_0^{1,p}(I)$ if and only if $\tilde{u} \in W_0^{1,p}(\mathbb{R})$.

Proof: $u \in W_0^{1,p}(I) \Longrightarrow$

$$|\int_I u\varphi'| = |-\int_I u'\varphi| \leq \|u'\|_{L^p}\|\varphi\|_{L^{p'}}, \quad \forall\, \varphi \in C_0^1(I)$$

where $1/p + 1/p^{'} = 1$.

Define the linear form on $C_0^1(I)$ by $F(\varphi) = -\int_i u\varphi'$.

It is clear that $\|F(\varphi)\| \le C\|\varphi\|_{L^{p'}} \Longrightarrow F$ is bounded on a subspace of $L^p(I)$. So it can be extended to Φ which is bounded by the same constant C on $L^{p'}(I)$. Riesz Representation Theorem implies the existence of $g \in L^p(I)$ such that

$$\Phi(\varphi) = \int_I g\varphi', \quad \forall\ \varphi \in L^{p'}(I).$$

Thus (by definition) $g = u' \Longrightarrow u \in W^{1,p}(I)$. To prove that $u \in W_0^{1,p}(I)$, we use the fact that

$$F(\varphi) = -\int_I u\varphi' = \int_I u'\varphi, \qquad \forall \varphi \in C^1(\bar{I}) \subset L^{p'}(I)$$

But

$$-\int_a^b u^{'}\varphi = -u(b)\varphi(b) + u(a)\varphi(a) + \int_a^b u\varphi^{'}$$

so

$$-u(b)\varphi(b) + u(a)\varphi(a) = 0, \qquad \forall \varphi \in C^1(\bar{I})$$

$\Longrightarrow u(a) = u(b) = 0 \Longrightarrow u \in W_0^{1,p}(I)$.

Note that we needed $1 < p < \infty$ for the reflexivity of $L^{p^{'}}(I)$ hence we could apply the Riesz representation theorem.

2) $u \in W^{1,p}(I) \Longrightarrow u \in L^p(\mathbb{R})$ since $\int_{\mathbb{R}} |\tilde{u}|^p = \int_I |u|^p$. Let $\varphi \in C_0^1(\mathbb{R})$. So

$$\int_{\mathbb{R}} \tilde{u}\varphi^{'} = \int_a^b u\varphi^{'} = u(b)\varphi(b) - u(a)\varphi(a) - \int_a^b u^{'}\varphi = -\int_a^b u^{'}\varphi$$

since $u(a) = u(b) = 0$. Thus

$$\int_{\mathbb{R}} \tilde{u}\varphi^{'} = -\int_a^b u^{'}\varphi = \int_{\mathbb{R}} \tilde{u}^{'}\varphi$$

where

$$\tilde{u}^{'}(x) = \begin{cases} u^{'}(x), & x \in I \\ 0, & x \in \mathbb{R}\backslash I \end{cases}$$

and $\tilde{u}^{'} \in L^p(\mathbb{R})$.

To show that $u \in W^{1,p}(I)$, we take $\varphi \in C_0^1(I)$ and compute

$$\int_I u\varphi' = \int_{\mathbb{R}} \tilde{u}\tilde{\varphi}^{'} = -\int_{\mathbb{R}}^{'} \tilde{u}\tilde{\varphi} = -\int_I g\varphi$$

where $\tilde{\varphi}$ is the extension of φ, which belongs to $C_0^1(\mathbb{R})$ since supp$\tilde{\varphi}$ = supp$\varphi \subset\subset \mathbb{R}$.

Next, since $\tilde{u} \in W^{1,p}(\mathbb{R})$ so $\tilde{u}$ is continuous, hence $u(a) = \tilde{u}(a) = u(b) = \tilde{u}(b) = 0$. Thus $u \in W_0^{1,p}(I)$.

Theorem: (Poincare$^{'}$s inequality). Suppose that I is bounded. Then there exists a constant $C = C(I) > 0$ such that

$$\int_I |u|^p \le C\int_I |u'|^p, \quad \forall\ u \in W_0^{1,p}(I).$$

Proof: For $u \in W_0^{1,p}(I)$, we have

$$u(x) = u(a) + \int_a^x u^{'}(t)dt = \int_a^x u^{'}(t)dt$$

so

$$\begin{aligned} |u(x)|^p &\leq \Big(\int_a^x |u^{'}(t)|dt\Big)^p \leq \Big[(x-a)^{1/p^{'}}(\int_a^b |u^{'}(t)|^p dt)^{1/p}\Big]^p \\ &\leq (x-a)^{p/p^{'}}||u^{'}||_p^p, \qquad \frac{1}{p}+\frac{1}{p^{'}} = 1 \end{aligned}$$

Integrate over(a,b) to get

$$\int |u(x)|^p \leq \frac{|I|^p}{p}||u^{'}||_p^p$$

hence

$$||u||_p \leq \frac{|I|}{(p)^{1/p}}\ ||u^{'}||_p$$

Remark: 1) From the above inequality it is important that I be bounded.

2) For $p = +\infty$, we have $C = |I|$, that is $\lim_{p\to\infty}(p)^{1/p} = 1$.

3) Looking carefully into the proof, we easily see that Poincare$^{'}$s inequality holds for $u \in W^{1,p}(I)$ with $u(c) = 0, \quad a \leq c \leq b$

Corollary: The quantity $\|u'\|_{L^p}$ define an equivalent norm on $W_0^{1,p}(I)$ and $< u, v >= \int_a^b u'v'$ define an equivalent inner product on $H_0^1(I)$.

3 The Space $W_0^{m,p}(I)$

Definition: Let $1 \leq p < +\infty$, we define $W_0^{m,p}(I)$ to be the closure of $C_0^m(I)$ (or $C_0^\infty(I)$) with respect to the norm of $W^{m,p}(I)$. We denote by $H_0^m(I) = W_0^{m,2}(I)$.

Remark: All the properties of $W_0^{1,p}(I)$ hold for $W_0^{m,p}(I)$.

Proposition: For $1 \leq p < \infty$,

$$W_0^{1,p}(I) = \{u \in W^{m,p}(I)\ /\ u_{|\partial I} = u^{'}_{|\partial I} = \cdots = u^{(m-1)}_{|\partial I} = 0\}$$

Remark: There are 2 important spaces, namely

$$W_0^{2,p}(I) = \{u \in W^{2,p}(I)\ /\ u_{|\partial I} = u^{'}_{|\partial I} = 0\}$$

and

$$W^{2,p}(I) \cap W_0^{1,p}(I) = \{u \in W^{2,p}(I)\ /\ u_{|\partial I} = 0\}$$

Example: On $I = (0,\pi)$, let $u(x) = \sin x$. It is clear that $u \in W^{m,p}(I), \forall\ m \geq 2,\ p \geq 1$ and $u(0) = u(\pi) = 0$; but $u'(0)u'(\pi) \neq 0$. So $u \notin W_0^{2,p}(I)$ however $u \in W^{2,p}(I) \cap W_0^{1,p}(I)$.

Exercise: Show that if $u \in W^{1,p}(I)$, for I bounded, and $\int_a^b u(x)dx = 0$ (zero mean) then Poincare$^{'}$s inequality holds.