1 Embedding

Definition: Given two sets A and B. We say that A is embedded in B; $A \subset B$; if $\forall\, a \in A$, $a \in B$.
Theorem: (Embedding Theorem). There exists a constant C, depending on $|I|$ such that

$$\|u\|_{L^\infty(I)} \leq C\|u\|_{W^{1,p}(I)}, \quad \forall\, u \in W^{1,p}(I).$$

Proof: Without loss of generality, we take $I = \mathbb{R}$; otherwise we extend u on $\mathbb{R}$. So, let $G(s)\ |s|^{p-1}s$, $p \geq 1$ and $w = G(v)$ for $v \in C_0^1(\mathbb{R})$. It is clear that $w \in C_0^1(\mathbb{R})$ and $w' = p|v|^{p-1}v'$. Thus

$$G(v(x)) = |v|^{p-1}v(x) = \int_{-\infty}^{x} p|v|^{p-1}v'(t)dt$$

$\Longrightarrow$

$$|v(x)|^p \leq p||v||_p^{p-1}||v'||_p \leq p||v||^p_{W^{1,p}(\mathbb{R})}, \qquad \forall x \in \mathbb{R}$$

Therefore

$$||v||_\infty \leq (p)^{1/p}\,||v||_{W^{1,p}(\mathbb{R})}.$$

$u \in W^{1,p}(\mathbb{R})$,we approximate u by a sequence $(u_n) \subset C_0^\infty(\mathbb{R})$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R})$. We also have $||u_n||_\infty \leq C||u_n||_{W^{1,p}}$. By letting $n \to \infty$, we obtain the desired result.
Remark: If $I = (\mathbb{R})$, the embedding constant is $C = (p)^{1/p}$. If $I \neq (\mathbb{R})$, the embedding constant is $C = C(|I|, p)$; this comes from the extension operator.
Definition: Given two metric spaces $X \subset Y$, We say that X is compactly embedded in Y if any bounded subset of X has a convergent sequence in Y.
Theorem: (The compact embedding theorem): Suppose that I is a bounded and open interval. Then

a) The embedding $W^{1,p}(I) \hookrightarrow C(\bar{I})$ is compact for $\rho > 1$.
b) The embedding $W^{1,1}(I) \hookrightarrow L^s(I)$ is compact for $s \in [1, +\infty)$.

Proof: *a*) Let B be the unit ball in $W^{1,p}(I)$, $B = \{u \in W^{1,p}(I)/\|u\|_{W^{1,p}} \leq 1\}$. For any x, y in I, we have

$$\begin{aligned} |u(x) - u(y)| &= |\int_x^y u'(t)dt| \leq ||u||_{W^{1,p}}|x-y|^{1/p'} \\ &\leq |x-y|^{1/p'}, \qquad (\frac{1}{p} + \frac{1}{p'}), \quad p > 1. \end{aligned}$$

So B is equicontinuous. Also, the previous theorem we have $||u||_\infty \leq C||u||_{W^{1,p}}$.Therefore B is uniformly bounded. Arzela-Ascoli shows that B is relatively compact.

b) To show that $W^{1,1}(I)$ is compactly imbedded in $L^s(I)$, $\quad s \geq 1$, we use a result of the L^p spaces. That is we show that $\|\tau_h u - u\|_{L^s(\omega)} \to 0$ uniformly in u as $h \to 0$, where $u \in B$ and $\omega \subset\subset I$..

$$\begin{aligned} \int_\omega |u(x+h) - u(x)|^s dx &= \int_\omega |u(x+h) - u(x)|^{s-1}|u(x+h) - u(x)|dx \\ &\leq 2||u||_\infty^{s-1} \int_\omega |u(x+h) - u(x)|dx \end{aligned}$$

$$
\begin{aligned}
&\leq 2C||u||_{W^{1,1}}^{s-1}\int_{\omega}\int_{x}^{x+h}|u'(t)|dtdx \\
&\leq 2C||u||_{W^{1,1}}^{s-1}||u'||_1|h| \leq 2C|h|
\end{aligned}
$$

We conclude then

$$\|\tau_h u - u\|_{L^s} \leq C'|h|^{1/s} \to 0$$

uniformly in u as $h \to 0$; .hence B is relatively compact in $L^s(I)$.
Remark: The application of Ascoli' s theorem requires that I be bounded.
Theorem: If I is an interval (bounded or not). Then $W^{1,p}(I) \subset L^q(I)$, $\forall q \in [p, \infty]$.
Proof: $q = p$ or $q = \infty$ is trivial. For $q \in (p, \infty)$, we have

$$\int_I |u|^q \leq ||u||_\infty^{q-p}\int_I |u|^p.$$

Remark: 1) If I bounded then :$q \in [1, \infty]$.

2) It is important that I is bounded in the compact embedding

Example: Let

$$u_n(x) = \begin{cases} 0, & 0 \leq x < n-1 \\ x - n + 1, & n-1 \leq x < n \\ 1, & n \leq x < n+1 \\ -x + n + 2, 1, & n+1 \leq x \leq n+2 \\ 0, & x > n+2 \end{cases}.$$

be defined and continuous on $I = (0, +\infty)$. It is easy to verify that

$$\int |u_n|^p \leq 3, \qquad \int |u_n^{'}|^p = 2, \qquad \forall p \geq 1$$

hence $\|u_n\|_{W^{1,p}}$ is bounded. Also $\lim_{n\to\infty} u_n(x) = u(x) = 0$. This is a simple convergence; however, for any subsequence (u_{n_k}) we have

$$\sup_{0<x<\infty} |u_{n_k}(x) - u(x)| = \sup_{0<x<\infty} |u_{n_k}(x)| = 1$$

Thus

$$\lim_{k\to\infty} \sup_{0<x<\infty} |u_{n_k}(x)| = 1 \neq 0.$$

Thus we cannot extract a subsequence, which converges to $u(x) \equiv 0$ in $C(I)$ uniformly.

Also, note that

$$\|u_{n_k} - u\|_{L^s}^s = \int_0^\infty |u_{n_k}|^s \geq \int_{n_k}^{n_{k+1}} |u_{n_k}|^s = 1$$

$\Longrightarrow$

$$\lim_{k\to\infty} \|u_{n_k} - u\|_{L^s}^s \geq 1, \qquad \forall s \geq 1.$$

So the embedding of $W^{1,1}(I)$ in $L^s(I)$, $\forall s \geq 1$, is not compact.
Remark: The embedding of $W^{1,1}(I)$ in $C(I)$ is continuous but is not necessarily

compact even if I is bounded.
Example: For $n \geq 2$, let

$$u_n(x) = \begin{cases} 1-nx, & 0 < x \leq 1/n \\ 0, & 1/n < x < 1 \end{cases}$$

be defined on $I=(0,1)$. We have

$$\int_0^1 |u(x)|dx = \frac{1}{2n} < 1, \qquad \int_0^1 |u'(x)|dx = \int_0^{1/n} ndx = 1$$

But

$$\int_0^1 |u'(x)|^p dx = \int_0^{1/n} n^p dx = n^{p-1}$$

So (u_n) is bounded in $W^{1,1}(I)$ only. $\lim_{k\to\infty} u_{n_k}(x) = u(x) = 0$ for any subsequence but

$$\sup_{0<x<1} |u_{n_k}(x) - u(x)| = \sup_{0<x<1} |u_{n_k}(x)| = 1$$

Hence u_{n_k} cannot converge uniformly to u.
Corollary: Let $I=(a,\infty)$ and $u \in W^{1,p}(I)$, $1 \leq p < \infty$. Then $\lim_{x\to\infty} u(x) = 0$
Proof: $u \in W^{1,p}(I)$, so there exits $(u_n) \subset C_0^\infty(\mathbb{R})$ such that $u_{n|I} \to u$ in $L^\infty(I) \Longrightarrow$

$$|u(x) \leq |u(x)-u_n(x)| + |u_n(x)| \leq \varepsilon + |u_n(x)|$$

for n large enough. So

$$\lim_{x\to\infty} |u(x) \leq \varepsilon + \lim_{x\to\infty} |u_n(x)| = \varepsilon$$

Since ε is arbitrary then $\lim_{x\to\infty} |u(x) = 0$.
Remark: 1) For $p=\infty$, the assertion of the corollary is not true. Take $u(x)=1$ for example.

2) The $W^{1,p}(I)$ functions do not oscillate at infinity. They are of bounded variations.
Corollary: If u and v are in $W^{1,p}(I)$, $\quad 1 \leq p \leq +\infty$, then uv in $W^{1,p}(I)$ and

$$(uv)' = u'v + uv' \qquad (*)$$

Moreover, for all x, y in $\bar{I}$, we have

$$\int_x^y (u'v+uv')(t)dt = u(y)v(y) - u(x)v(x) \qquad .(**)$$

Proof: $u,v \in W^{1,p}(I) \Longrightarrow u,v \in L^\infty(I) \Longrightarrow uv \in L^p(I)$ since $\int_I |uv|^p dx \leq \|u\|_\infty^p \|v\|_{L^p}^p$.
Case 1. $1 \leq p < \infty$.

Let (u_n) and (v_n) be two sequences in $C_0^\infty(\mathbb{R})$ such that

$u_{n|I} \to u$, $v_{n|I} \to v$ in $W^{1,p}(I)$ (hence in $L^\infty(I)$). Therefore $u_n v_n \to uv$ in $L^\infty(I)$. We also have

$$(u_n v_n)' = u_n' v_n + u_n v_n' \to u'v + uv' \text{ in } L^p(I).$$

$\Longrightarrow (uv)' = u'v + uv' \in L^p(I)$; hence $uv \in W^{1,p}(I)$. By integrating over (x, y):

$$\int_x^y (u_n v_n)' = \int_x^y (u_n' v_n + u_n v_n') = u_n(y)v_n(y) - u_n(x)v_n(x).$$

By letting $n \to \infty$, we obtain $(**)$.

Case 2. $p = +\infty$

$u, v \in W^{1,\infty}(I) \Longrightarrow uv,\ u'v + uv' \in L^\infty(I)$. We have to verify that $(uv)' = uv' + u'v \in L^\infty(I)$. Let $\varphi \in C_0^1(I)$; so for J bounded and $\operatorname{supp} \varphi \subset J \subset\subset I$, we have u and $v \in L^q(J), \quad \forall\, q < \infty$ and consequently, by Case 1, we obtain

$$\int_I uv\varphi' = \int_J uv\varphi' = -\int_J (u'v + uv')\varphi = -\int_I (u'v + uv')\varphi$$

Thus

$$(uv)' = uv' + uv' \quad in \quad L^\infty(I).$$

This completes the proof

Corollary: Let $G \in C^1(\mathbb{R})$, such that $G(0) = 0$ and $u \in W^{1,p}(I)$. Then $G \circ u \in W^{1,p}(I)$ and $(G \circ u)' = (G' \circ u)u'$.

Proof: Let $u \in W^{1,p}(I) \Longrightarrow$ there exists $M > 0$ such that $-M \leq u(x) \leq M, \quad \forall\, x \in I$; $u \in C(\bar{I})$. G' is continuous and $G(0) = 0 \Longrightarrow |G(s)| \leq C|s|, \quad \forall\, s \in [-M, M] \Longrightarrow G \circ u \in L^p(I)$. Also $(G' \circ u)u' \in L^p(I)$, since $G' \circ u \in L^\infty$ and $u' \in L^p$.

Now we should verify that

$$\int_I (G \circ u)\varphi' = -\int (G' \circ u)u'\varphi, \qquad \forall \varphi \in C_0^1(I)$$

Case 1. $1 \leq p < +\infty$.

There exists $(u_n) \in C_0^\infty(\mathbb{R})$ such that $u_{n|I} \to u$ in $W^{1,p}(I)$ (hence in $L^\infty(I)$). So $G \circ u_n \to G \circ u$ and $(G' \circ u_n)u_n' \to (G' \circ u)u'$ in $L^p(I)$ and

$$\int (G \circ u_n)\varphi' = -\int (G' \circ u_n)u_n'\varphi, \quad \forall\, \varphi \in C_0^\infty(I).$$

By taking n to ∞, we obtain

$$\int_I (G \circ u)\varphi' = -\int (G' \circ u)u'\varphi, \quad \forall\, \varphi \in C_0^\infty(I).$$

Therefore $(G \circ u)' = (G' \circ u)u'$ by definition of weak derivative.

Case 2. $p = \infty$

We repeat the same analysis of the previous corollary.

Remark: The condition $G(0) = 0$ is not necessary when I is bounded.

Remark: $W^{1,p}(I)$ is called Banach algebra since $uv \in W^{1,p}(I)$ whenever u and v are in $W^{1,p}(I)$. This is not the case for $L^p(I)$ even for I bounded.

Example: $u(x) = 1/\sqrt{x} \in L^1((0,1))$ since

$$\int_0^1 1/\sqrt{x}dx = 21/\sqrt{x}|_0^1 = 2.$$

$u' \not\in L^1(0,1)$ since

$$\int_0^1 u'(x)dx = u(x)r|_0^1 = \infty$$

For $v = u$, we have $uv = 1/x$,

$$\int_0^1 (uv)dx = \log x|_0^1 = \infty \Longrightarrow uv \not\in L^1((0,1)).$$

Remark: When I is unbounded $G(0) = 0$ is essential
Example: Let $I = (0, +\infty)$ and $u \in L^p(I), \quad \forall\ 1 \leq p < \infty$. Take $G(s) = a \neq 0$; hence $G(0) \neq 0$. Note that $G \circ u = a \not\in L^p(I)$.