

1 Embedding

Definition: Given two sets A and B . We say that A is embedded in B ; $A \subset B$; if $\forall a \in A, a \in B$.

Theorem: (Embedding Theorem). There exists a constant C , depending on $|I|$ such that

$$\|u\|_{L^\infty(I)} \leq C\|u\|_{W^{1,p}(I)}, \quad \forall u \in W^{1,p}(I).$$

Proof: Without loss of generality, we take $I = \mathbb{R}$; otherwise we extend u on \mathbb{R} . So, let $G(s) = |s|^{p-1}s$, $p \geq 1$ and $w = G(v)$ for $v \in C_0^1(\mathbb{R})$. It is clear that $w \in C_0^1(\mathbb{R})$ and $w' = p|v|^{p-1}v'$. Thus

$$G(v(x)) = |v|^{p-1}v(x) = \int_{-\infty}^x p|v|^{p-1}v'(t)dt$$

\implies

$$|v(x)|^p \leq p\|v\|_p^{p-1}\|v'\|_p \leq p\|v\|_{W^{1,p}(\mathbb{R})}^p, \quad \forall x \in \mathbb{R}$$

Therefore

$$\|v\|_\infty \leq (p)^{1/p} \|v\|_{W^{1,p}(\mathbb{R})}.$$

$u \in W^{1,p}(\mathbb{R})$, we approximate u by a sequence $(u_n) \subset C_0^\infty(\mathbb{R})$ such that $u_n \rightarrow u$ in $W^{1,p}(\mathbb{R})$. We also have $\|u_n\|_\infty \leq C\|u_n\|_{W^{1,p}}$. By letting $n \rightarrow \infty$, we obtain the desired result.

Remark: If $I = (\mathbb{R})$, the embedding constant is $C = (p)^{1/p}$. If $I \neq (\mathbb{R})$, the embedding constant is $C = C(|I|, p)$; this comes from the extension operator.

Definition: Given two metric spaces $X \subset Y$, We say that X is compactly embedded in Y if any bounded subset of X has a convergent sequence in Y .

Theorem: (The compact embedding theorem): Suppose that I is a bounded and open interval. Then

- a) The embedding $W^{1,p}(I) \hookrightarrow C(\bar{I})$ is compact for $p > 1$.
- b) The embedding $W^{1,1}(I) \hookrightarrow L^s(I)$ is compact for $s \in [1, +\infty)$.

Proof: a) Let B be the unit ball in $W^{1,p}(I)$, $B = \{u \in W^{1,p}(I) / \|u\|_{W^{1,p}} \leq 1\}$. For any x, y in I , we have

$$\begin{aligned} |u(x) - u(y)| &= \left| \int_x^y u'(t)dt \right| \leq \|u\|_{W^{1,p}} |x - y|^{1/p'} \\ &\leq |x - y|^{1/p'}, \quad \left(\frac{1}{p} + \frac{1}{p'}\right), \quad p > 1. \end{aligned}$$

So B is equicontinuous. Also, the previous theorem we have $\|u\|_\infty \leq C\|u\|_{W^{1,p}}$. Therefore B is uniformly bounded. Arzela-Ascoli shows that B is relatively compact.

b) To show that $W^{1,1}(I)$ is compactly imbedded in $L^s(I)$, $s \geq 1$, we use a result of the L^p spaces. That is we show that $\|\tau_h u - u\|_{L^s(\omega)} \rightarrow 0$ uniformly in u as $h \rightarrow 0$, where $u \in B$ and $\omega \subset\subset I$.

$$\begin{aligned} \int_\omega |u(x+h) - u(x)|^s dx &= \int_\omega |u(x+h) - u(x)|^{s-1} |u(x+h) - u(x)| dx \\ &\leq 2\|u\|_\infty^{s-1} \int_\omega |u(x+h) - u(x)| dx \end{aligned}$$

$$\begin{aligned}
&\leq 2C \|u\|_{W^{1,1}}^{s-1} \int_{\omega} \int_x^{x+h} |u'(t)| dt dx \\
&\leq 2C \|u\|_{W^{1,1}}^{s-1} \|u'\|_1 |h| \leq 2C |h|
\end{aligned}$$

We conclude then

$$\|\tau_h u - u\|_{L^s} \leq C' |h|^{1/s} \rightarrow 0$$

uniformly in u as $h \rightarrow 0$; hence B is relatively compact in $L^s(I)$.

Remark: The application of Ascoli's theorem requires that I be bounded.

Theorem: If I is an interval (bounded or not). Then $W^{1,p}(I) \subset L^q(I)$, $\forall q \in [p, \infty]$.

Proof: $q = p$ or $q = \infty$ is trivial. For $q \in (p, \infty)$, we have

$$\int_I |u|^q \leq \|u\|_{\infty}^{q-p} \int_I |u|^p.$$

Remark: 1) If I bounded then $q \in [1, \infty]$.

2) It is important that I is bounded in the compact embedding

Example: Let

$$u_n(x) = \begin{cases} 0, & 0 \leq x < n-1 \\ x-n+1, & n-1 \leq x < n \\ 1, & n \leq x < n+1 \\ -x+n+2, & n+1 \leq x \leq n+2 \\ 0, & x > n+2 \end{cases}.$$

be defined and continuous on $I = (0, +\infty)$. It is easy to verify that

$$\int |u_n|^p \leq 3, \quad \int |u_n'|^p = 2, \quad \forall p \geq 1$$

hence $\|u_n\|_{W^{1,p}}$ is bounded. Also $\lim_{n \rightarrow \infty} u_n(x) = u(x) = 0$. This is a simple convergence; however, for any subsequence (u_{n_k}) we have

$$\sup_{0 < x < \infty} |u_{n_k}(x) - u(x)| = \sup_{0 < x < \infty} |u_{n_k}(x)| = 1$$

Thus

$$\lim_{k \rightarrow \infty} \sup_{0 < x < \infty} |u_{n_k}(x)| = 1 \neq 0.$$

Thus we cannot extract a subsequence, which converges to $u(x) \equiv 0$ in $C(I)$ uniformly.

Also, note that

$$\|u_{n_k} - u\|_{L^s}^s = \int_0^{\infty} |u_{n_k}|^s \geq \int_{n_k}^{n_k+1} |u_{n_k}|^s = 1$$

\implies

$$\lim_{k \rightarrow \infty} \|u_{n_k} - u\|_{L^s}^s \geq 1, \quad \forall s \geq 1.$$

So the embedding of $W^{1,1}(I)$ in $L^s(I)$, $\forall s \geq 1$, is not compact.

Remark: The embedding of $W^{1,1}(I)$ in $C(I)$ is continuous but is not necessarily

compact even if I is bounded.

Example: For $n \geq 2$, let

$$u_n(x) = \begin{cases} 1 - nx, & 0 < x \leq 1/n \\ 0, & 1/n < x < 1 \end{cases}$$

be defined on $I = (0, 1)$. We have

$$\int_0^1 |u(x)| dx = \frac{1}{2n} < 1, \quad \int_0^1 |u'(x)| dx = \int_0^{1/n} n dx = 1$$

But

$$\int_0^1 |u'(x)|^p dx = \int_0^{1/n} n^p dx = n^{p-1}$$

So (u_n) is bounded in $W^{1,1}(I)$ only. $\lim_{k \rightarrow \infty} u_{n_k}(x) = u(x) = 0$ for any subsequence but

$$\sup_{0 < x < 1} |u_{n_k}(x) - u(x)| = \sup_{0 < x < 1} |u_{n_k}(x)| = 1$$

Hence u_{n_k} cannot converge uniformly to u .

Corollary: Let $I = (a, \infty)$ and $u \in W^{1,p}(I)$, $1 \leq p < \infty$. Then $\lim_{x \rightarrow \infty} u(x) = 0$

Proof: $u \in W^{1,p}(I)$, so there exists $(u_n) \subset C_0^\infty(\mathbb{R})$ such that $u_n|_I \rightarrow u$ in $L^\infty(I) \implies$

$$|u(x)| \leq |u(x) - u_n(x)| + |u_n(x)| \leq \varepsilon + |u_n(x)|$$

for n large enough. So

$$\lim_{x \rightarrow \infty} |u(x)| \leq \varepsilon + \lim_{x \rightarrow \infty} |u_n(x)| = \varepsilon$$

Since ε is arbitrary then $\lim_{x \rightarrow \infty} |u(x)| = 0$.

Remark: 1) For $p = \infty$, the assertion of the corollary is not true. Take $u(x) = 1$ for example.

2) The $W^{1,p}(I)$ functions do not oscillate at infinity. They are of bounded variations.

Corollary: If u and v are in $W^{1,p}(I)$, $1 \leq p \leq +\infty$, then uv in $W^{1,p}(I)$ and

$$(uv)' = u'v + uv' \quad (*)$$

Moreover, for all x, y in \bar{I} , we have

$$\int_x^y (u'v + uv')(t) dt = u(y)v(y) - u(x)v(x) \quad (**)$$

Proof: $u, v \in W^{1,p}(I) \implies u, v \in L^\infty(I) \implies uv \in L^p(I)$ since $\int_I |uv|^p dx \leq \|u\|_\infty^p \|v\|_{L^p}^p$.

Case 1. $1 \leq p < \infty$.

Let (u_n) and (v_n) be two sequences in $C_0^\infty(\mathbb{R})$ such that

$u_n|_I \rightarrow u, v_n|_I \rightarrow v$ in $W^{1,p}(I)$ (hence in $L^\infty(I)$). Therefore $u_n v_n \rightarrow uv$ in $L^\infty(I)$.

We also have

$$(u_n v_n)' = u_n' v_n + u_n v_n' \rightarrow u'v + uv' \text{ in } L^p(I).$$

$\implies (uv)' = u'v + uv' \in L^p(I)$; hence $uv \in W^{1,p}(I)$. By integrating over (x, y) :

$$\int_x^y (u_n v_n)' = \int_x^y (u_n' v_n + u_n v_n') = u_n(y)v_n(y) - u_n(x)v_n(x).$$

By letting $n \rightarrow \infty$, we obtain (**).

Case 2. $p = +\infty$

$u, v \in W^{1,\infty}(I) \implies uv, u'v + uv' \in L^\infty(I)$. We have to verify that $(uv)' = uv' + u'v \in L^\infty(I)$. Let $\varphi \in C_0^1(I)$; so for J bounded and $\text{supp } \varphi \subset J \subset\subset I$, we have u and $v \in L^q(J)$, $\forall q < \infty$ and consequently, by Case 1, we obtain

$$\int_I uv\varphi' = \int_J uv\varphi' = - \int_J (u'v + uv')\varphi = - \int_I (u'v + uv')\varphi$$

Thus

$$(uv)' = uv' + uv' \quad \text{in } L^\infty(I).$$

This completes the proof

Corollary: Let $G \in C^1(\mathbb{R})$, such that $G(0) = 0$ and $u \in W^{1,p}(I)$. Then $G \circ u \in W^{1,p}(I)$ and $(G \circ u)' = (G' \circ u)u'$.

Proof: Let $u \in W^{1,p}(I) \implies$ there exists $M > 0$ such that $-M \leq u(x) \leq M$, $\forall x \in I$; $u \in C(\bar{I})$. G' is continuous and $G(0) = 0 \implies |G(s)| \leq C|s|$, $\forall s \in [-M, M] \implies G \circ u \in L^p(I)$. Also $(G' \circ u)u' \in L^p(I)$, since $G' \circ u \in L^\infty$ and $u' \in L^p$.

Now we should verify that

$$\int_I (G \circ u)\varphi' = - \int_I (G' \circ u)u'\varphi, \quad \forall \varphi \in C_0^1(I)$$

Case 1. $1 \leq p < +\infty$.

There exists $(u_n) \in C_0^\infty(\mathbb{R})$ such that $u_n|_I \rightarrow u$ in $W^{1,p}(I)$ (hence in $L^\infty(I)$). So $G \circ u_n \rightarrow G \circ u$ and $(G' \circ u_n)u_n' \rightarrow (G' \circ u)u'$ in $L^p(I)$ and

$$\int_I (G \circ u_n)\varphi' = - \int_I (G' \circ u_n)u_n'\varphi, \quad \forall \varphi \in C_0^\infty(I).$$

By taking n to ∞ , we obtain

$$\int_I (G \circ u)\varphi' = - \int_I (G' \circ u)u'\varphi, \quad \forall \varphi \in C_0^\infty(I).$$

Therefore $(G \circ u)' = (G' \circ u)u'$ by definition of weak derivative.

Case 2. $p = \infty$

We repeat the same analysis of the previous corollary.

Remark: The condition $G(0) = 0$ is not necessary when I is bounded.

Remark: $W^{1,p}(I)$ is called Banach algebra since $uv \in W^{1,p}(I)$ whenever u and v are in $W^{1,p}(I)$. This is not the case for $L^p(I)$ even for I bounded.

Example: $u(x) = 1/\sqrt{x} \in L^1((0, 1))$ since

$$\int_0^1 1/\sqrt{x} dx = 21/\sqrt{x}|_0^1 = 2.$$

$u' \notin L^1(0, 1)$ since

$$\int_0^1 u'(x) dx = u(x)r|_0^1 = \infty$$

For $v = u$, we have $uv = 1/x$,

$$\int_0^1 (uv) dx = \log x|_0^1 = \infty \implies uv \notin L^1((0, 1)).$$

Remark: When I is unbounded $G(0) = 0$ is essential

Example: Let $I = (0, +\infty)$ and $u \in L^p(I)$, $\forall 1 \leq p < \infty$. Take $G(s) = a \neq 0$; hence $G(0) \neq 0$. Note that $G \circ u = a \notin L^p(I)$.