# 1 Regularity of weak solutions

**Definition**. Let $\Omega$ be an open set of $\mathbb{R}^N$. We say that $\Omega$ is of class $C^m$, if for each $x \in \Gamma$, there exists a neighborhood $U$ of $x$ in $\mathbb{R}^N$ and a bijective application $H : Q \to U$ such that

$$\begin{aligned} H \in C^m(\bar{Q}), \qquad H^{-1} \in C^m(\bar{U}), \\ H(Q_+) = U \cap \Omega, \qquad H(Q_0) = U \cap \Gamma. \end{aligned}$$

$\Omega$ is said to be of class $C^\infty$, if it is of class $C^m, \ \forall m \geq 1$.

Here are some regularity results.

**Theorem 1**. Suppose that $f \in L^2(\mathbb{R}^N)$ and $u \in H^1(\mathbb{R}^N)$ such that

$$\int (\nabla u \cdot \nabla \phi + u\phi) = \int f\phi, \quad \forall \phi \in H^1(\mathbb{R}^N). \tag{1}$$

then

$$u \in H^2(\mathbb{R}^N) \text{ and } \|u\|_{H^2} \leq c\|f\|_{L^2}.$$

**Proof**. For $h \in \mathbb{R}^N, h \neq 0$, we set

$$D_h u = \frac{1}{|h|}(\tau_h u - u),$$

that is

$$D_h u(x) = \frac{u(x+h) - u(x)}{|h|}$$

Let $\phi = D_{-h}(D_h u)$. It is clear that $\phi \in H^1(\mathbb{R}^N)$. Since $u \in H^1(\mathbb{R}^N)$ and use it for (1) to obtain

$$\int |\nabla(D_h u)|^2 + \int |D_h u|^2 = \int f(D_{-h}(Du))$$

which implies

$$\|D_h u\|_{H^1}^2 \leq \|f\|_{L^2}\|D_{-h}(Du)\|_{L^2} \tag{2}$$

In the other hand, we have

$$\|D_{-h}(Du)\|_{L^2} \leq \|\nabla(Du)\|_{L^2} \tag{3}$$

since

$$\|D_{-h}v\|_{L^2(\omega)} \leq \|\nabla v\|_{L^2(\mathbb{R}^N)}, \quad \forall v \in H^1 \text{ and } w \subset\subset \mathbb{R}^N.$$

Combining (2) and (3) we easily get

$$\|D_h u\|_{H^1} \leq \|f\|_{L_2}$$

In particular, we have

$$\|D_h \frac{\partial u}{\partial x_i}\|_{L_2} \leq \|f\|_{L_2}, \quad \forall i = 1, 2, \ldots, N.$$

S0,

$$\frac{\partial u}{\partial x_i} \in H^1(\mathbb{R}^N), \quad \forall i = 1, 2, \ldots, N$$

Hence $u \in H^2(\mathbb{R}^N)$.
**More regularity**
**Corollary 1.** If $f \in H^1(\mathbb{R}^N)$ and $u$ satisfies (1) then $u \in H^3(\mathbb{R}^N)$.

**Proof**. Let $\phi \epsilon C_0^\infty(\mathbb{R}^N)$; so $\frac{\partial \phi}{\partial x_i} \in C_0^\infty(\mathbb{R}^N), \ \forall i = 1, 2, \ldots N$. Since $u \in H^1$ (in fact $u \epsilon H^2$) then we have

$$\int \nabla u \cdot \nabla \left( \frac{\partial \phi}{\partial x_i} \right) + \int u \frac{\partial \phi}{\partial x_i} = \int f \frac{\partial \phi}{\partial x_i}, i = 1, 2, \ldots N.$$

By integrating we obtain

$$\int \nabla \left( \frac{\partial u}{\partial u_i} \right) \cdot \nabla \phi = \int \frac{\partial u}{\partial x_i} \phi = \int \frac{\partial f}{\partial x_i} \phi, \quad \forall \epsilon C_0^\infty(\mathbb{R}^N),$$

which implies that $\frac{\partial u}{\partial x_i} \in H^2, \ \forall i = 1, 2, \ldots N$ ; hence $u \in H^3(\mathbb{R}^N)$.

By repeating the same procedure we have the following:
**Corollary 2**. If $f \in H^m(\mathbb{R}^N)$ and $u$ satisfies (1). Then $u \in H^{m+2}(\mathbb{R}^N)$.

## 1.1 Case $\Omega = \mathbb{R}_+^N$.

**Reminder**. $\mathbb{R}_+^N = \{(x_1, x_2, \ldots, x_{N-1}\ x_N), \quad x_N \geq 0\}$
**Definition**. We say that $h // \Gamma$ if $h \in \mathbb{R}^{N-1} \times \{0\}$ i.e. $h = (h_1, \ldots, h_{N-1}, 0)$.
**Lemma**. Suppose that $u \in H_0^1(\Omega)$ and $h // \Gamma$ then $D_h u \in H_0^1(\Omega)$.
**Proposition**. Let $f \in L^2(\Omega)$ and suppose that $u \in H_0^1(\Omega)$ satisfies,

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega u \phi = \int f \phi, \quad \forall \phi \in H_0^1(\Omega) \tag{4}$$

Then $u \in H^2(\Omega)$.

**Proof**. Let $h//\Gamma$ and use $\phi = D_{-h}(Du)$ in (4). Then

$$\int_\Omega |\nabla(D_h u)|^2 + \int_\Omega |D_h u|^2 = \int_\Omega f D_{-h}(D_h u)$$

So

$$\|D_h u\|_{H^1}^2 \leq \|f\|_{L^2} \|D_{-h}(D_h u)\|_{L^2}. \tag{5}$$

We then use the fact that

$$\|D_h v\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)}, \quad \forall v \in H^1(\Omega), \quad \forall h//\Gamma \tag{6}$$

to obtain from (5)

$$\|D_h u\|_{H^1} \leq \|f\|_{L^2}, \quad \forall h//\Gamma. \tag{7}$$

**Exercise**. Establish (6).

Let $1 \leq j \leq N,\ 1 \leq k \leq N-1$ and take $h = |h|e_k$. So, for $\phi \in C_0^\infty(\Omega)$, we have

$$\int_\omega D_h \left(\frac{\partial u}{\partial x_j}\right) \phi = -\int_\Omega u D_{-h} \left(\frac{\partial \phi}{\partial x_j}\right) \tag{8}$$

**Exercise**. Show (8).

Combining (7) and (8) we have

$$\left| \int_\Omega u D_{-h} \left(\frac{\partial \phi}{\partial x_j}\right) \right| \leq \|f\|_{L^2} \|\phi\|_{L^2},\ \forall 1 \leq j \leq N, \quad 1 \leq k \leq N-1.$$

As $h \to 0$, we obtain

$$\left| \int u \frac{\partial^2 \phi}{\partial x_k \partial x_j} \right| \leq \|f\|_{L^2} \|\phi\|_{L^2}. \tag{9}$$

Next, we show that

$$\left| \int u \frac{\partial^2 \phi}{\partial x_N^2} \right| \leq C \|f\|_{L^2} \|\phi\|_{L^2}, \quad \forall \phi \in C_0^\infty(\Omega).$$

To do this, we use (4). So, we get

$$\begin{aligned} \left| \int_\Omega u \frac{\partial^2 \phi}{\partial x_N^2} \right| &\leq \sum_{i=1}^{N-1} \left| \int_\Omega u \frac{\partial^2 \phi}{\partial x_i^2} \right| + \left| \int_\Omega (f-u)\phi \right| \\ &\leq C \|f\|_{L^2} \|\phi\|_{L^2}, \quad \forall \phi \in C_0^\infty(\Omega). \end{aligned}$$

We conclude that

$$\left| \int u \frac{\partial^2 \phi}{\partial x_j \partial x_k} \right| \leq C\|f\|_{L^2}\|\phi\|_{L^2}, \ \forall \phi \ \in \ C_0^\infty(\Omega) \text{ and } \forall 1 \leq j,k \leq N. \tag{10}$$

Consequently, $u \ \in \ H^2(\Omega)$.
**Remark**. In fact, (10) shows that there exist $g_{jk} \in L^2(\Omega)$ such that

$$\int_\Omega u \frac{\partial^2 \phi}{\partial x_j \partial x_k} = \int g_{jk}\phi, \ \forall \phi \ \in \ C_0^\infty(\Omega).$$

By using Hahn-Banach theorem, the desired result is established
**More regularity**
**Lemma 2**. Let $u \ \in \ H^2(\Omega) \cap H_0^1(\Omega)$ satisfy (4). Then

$$\frac{\partial u}{\partial x_j} \ \in \ H_0^1(\Omega), \ \ \forall j = 1, 2, \dots, N.$$

Moreover, we have

$$\int_\Omega \nabla \left( \frac{\partial u}{\partial x_j} \right) \cdot \nabla \phi + \int_\Omega \frac{\partial u}{\partial x_j} \phi = \int \frac{\partial f}{\partial x_j} \phi, \ \ \forall \phi \ \in \ H_0^1(\Omega). \tag{11}$$

**Proof**. Let $h = |h|e_j$, $1 \leq j \leq N-1$; then $D_h u \ \in \ H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is invariant under tangential translation.

From (6), we deduce that

$$\|D_h u\|_{H^1} \leq \|u\|_{H^2} \leq C\|f\|_{L^2}. \tag{12}$$

So there exists a sequence $h_n \to 0$ such that

$$D_{h_n} u \rightharpoonup g_j \text{ in } H_0^1(\Omega).$$

By using

$$\int_\Omega (D_{h_n} u)\phi = -\int u D_{-h}\phi, \ \ \forall \phi \ \in \ C_0^\infty(\Omega)$$

and letting $h_n \to 0$, we arrive at

$$\int g_j \phi = -\int u \frac{\partial \phi}{\partial x_j}, \ \forall \phi \ \in \ C_0^\infty(\Omega).$$

hence

$$\frac{\partial u}{\partial x_j} = g_j \ \in \ H_0^1|(\Omega).$$

To obtain (11), it suffices to use $\frac{\partial \phi}{\partial x_j}$ in (4) instead of $\phi \in C_0^\infty(\Omega)$.
**Exercise**. Verify (12)
**Corollary**. Suppose that $u \in H_0^1(\Omega)$ satisfies (4) and $f \in H^m(\Omega)$. Then $u \in H^{m+2}(\Omega)$.
**Proof**. From (11) and proposition 2, we obtain that

$$\frac{\partial u}{\partial x_i} \in H^2(\Omega) \cap H_0^1(\Omega), \ \forall i = 1, 2, \dots, N.$$

Consequently, $u \in H^3(\Omega)$.

By repeating the same procedure, se easily prove the corollary by induction.

## 1.2 General case

**Theorem.** Suppose that $\Omega$ is open and of class $C^2$, with $\Gamma$ bounded. let $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega u\phi = \int_\Omega f\phi, \quad \forall \phi \in H_0^1(\Omega). \tag{13}$$

Then $u \in H^2(\Omega)$ and $\|u\|_{H^2} \leq C\|f\|_{L^2}$, where $C$ is a constant depending on $\Omega$ only.

Moreover, if $\Omega$ is of class $C^{m+2}$ and $f \in H^m(\Omega)$. Then

$$u \in H^{m+2}(\Omega), \qquad \text{and} \|u\|_{m+2} \leq C\|f\|_m.$$

In particular, if $m > \frac{N}{2}$ then $u \in C^2(\bar{\Omega})$.

Finally, if $\Omega$ is of class $C^\infty$ and $f \in C^\infty(\bar{\Omega})$ then $u \in C^\infty(\bar{\Omega})$.
**Proof**. Involves the partition of unity, investigation of regularity in the interior of $\Omega$ and near $\Gamma$.