Global Existence and Asymptotic Behavior for a Nonlinear Viscoelastic Problem

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Abstract

In this paper the nonlinear viscoelastic wave equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma \Delta u_t = b|u|^{p-2}u$$

is considered. Using the potential well method a global existence and an exponential decay result are proved. Moreover, for sufficiently large values of the initial data and for a suitable relation between p and the relaxation function g, we establish an unboundedness result.

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1 Introduction

We consider the following initial boundary value problem

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma \Delta u_t = b|u|^{p-2}u, & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \ x \in \partial\Omega \ , t \geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$. The constants γ, ρ and b are positive and $p > 2$. The function $g(t)$ is positive and satisfies some conditions to be specified later. This type of equations appears in the models of nonlinear viscoelasticity (see [1], [2], and [10]). It also can be considered as a system governing the longitudinal motion of a viscoelastic configuration obeying a nonlinear Voight model (see [10] and [21]).

In the case $b = 0$, that is in the absence of the source term, this problem has been studied by Cavalcanti *et al.* in [4]. Assuming $0 < \rho \leq 2/(n-2)$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$, the authors proved a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. This latter result has been extended to the case $\gamma = 0$, that is without any mechanical dissipation, by the present authors in [18]. A related problem to (1.1) is

$$\begin{cases} u_{tt} - \Delta u + \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau + au_t = b|u|^{p-2}u, \quad \text{in } \Omega \times (0, \infty) \\ u(x,t) = 0, \ x \in \partial\Omega \ , t \geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{1.2}$$

which has been extensively studied. For example, in the presence of the viscoelastic term ($g \neq 0$), Cavalcanti *et al.* [5] studied (2) for a localized damping $a(x)u_t$ ($a(x)$ can be null on a part of the domain) and $b < 0$. They obtained an exponential rate of decay by assuming that the kernel g decays exponentially. This work extended the result of Zuazua [22], in which the author considered (1.2) with $g = 0$ and the linear damping is localized. When the damping is caused only by the memory term ($a = 0$) and in the absence of the source term, an exponential decay result can be established, at least for small initial data, by following the idea of proof of Muñoz Rivera [19], in which he proved that the first and the second-order energy functionals of the solution to a viscoelastic plate, decay exponentially provided that the kernel of the memory decays also exponentially (see also [6]). For nonexistence results, Messaoudi [17] considered (1.2) for $b > 0$ and with a nonlinear mechanical damping of the form $au_t|u_t|^{m-2}$ instead of au_t, and proved a blow up result for solutions with negative initial energy.

In the absence of the viscoelastic term ($g = 0$), it is well known that, for $a = 0$, the source term $bu|u|^{p-2}$ in (2) causes finite time blow up of solutions with negative initial energy (see [3]) and for $b = 0$, the damping term au_t assures global existence for arbitrary initial data (see [8], [11]). The interaction between the damping and the source terms was first considered by Levine in [12], [13]. In these papers, the author showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [7] extended Levine's result to the nonlinear damping case, where the linear damping is replaced by a nonlinear one of the form $au_t|u_t|^{m-2}$. In their work, the authors introduced a different method and determined suitable relations between m and p, for which there is global existence or alternatively finite time blow up. Precisely; they showed that solutions with negative energy continue to exist globally 'in time' if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is *sufficiently* negative. Without imposing the condition that the initial energy is sufficiently negative, Messaoudi [16] extended the blow up result of [7] to solutions with negative initial energy only. For more results of the same nature, we refer the reader to Kalantarov and Ladyzhenskaya [9], Levine and Serrin [14], Levine, Park, and Serrin [15] and Vitillaro [20].

In our case, the dissipations compete with the source term and it is interesting to study this interaction. We prove that, when the initial data are in a *stable* set, we have global existence. To achieve our goal, we use the potential well method.

Moreover, we will show that the solution goes to zero in an exponential rate provided that the relaxation function is also exponentially decaying to zero. Furthermore, we prove that, for large initial data and for a p satisfying a condition related to the relaxation function g, the solution grows up exponentially.

The paper is organized as follows. In Section 2, We present some notations and material needed for our work and we state, without a proof, a standard local existence theorem. Section 3 contains the statements and the proofs of the global existence and exponential decay results. The last section is devoted to the statement and the proof of the exponential growth result.

2 Preliminaries

In this section, we shall prepare some material needed in the proofs of our results. Namely, we introduce some notations and show the invariance of the set of initial data.

We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev spaces $H_0^1(\Omega)$ with their usual scalar products and norms. The symbols ∇ and Δ will stand for the gradient and the Laplacian, respectively. The prime $'$ and the subscript t will denote time differentiation.

For the relaxation function $g(t)$ we assume

(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1 function such that

$$1 - \int_0^\infty g(s)ds = l > 0.$$

(G2) There exists a positive constant m such that

$$g(t) \leq -mg'(t), \ t \geq 0.$$

Theorem 2.1 *Let $u_0, u_1 \in H_0^1(\Omega)$ be given and $\gamma \geq 0$. Assume that g* satisfies $(G1)$. Assume further that

$$\begin{aligned} 2 &\leq p \leq \frac{2(n-1)}{n-2}, \qquad n \geq 3 \\ p &\geq 2, \qquad n = 1, 2. \end{aligned} \tag{2.1}$$

Then problem (1.1) *has a unique local solution*

$$u, u_t \in C\left([0, \ T_m); H_0^1(\Omega)\right), \tag{2.2}$$

for some $T_m > 0$.

Remark 2.1 This theorem can be easily established by combining the arguments in [4] and [7].

Remark 2.2 Condition (2.1) is needed to establish the local existence result (see [3]). In fact under this condition, the nonlinearity is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$. Condition (G1) is necessary to guarantee the hyperbolicity of the system (1.1).

Next, we introduce

$$I(t) = I(u, u_t) := \left(1 - \int_0^t g(s)ds\right) ||\nabla u||_2^2 + ||\nabla u_t||_2^2 + (g \circ \nabla u)(t) - b||u||_p^p,$$

$$J(t) = J(u, u_t) := \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) ||\nabla u(t)||_2^2 + \frac{1}{2}||\nabla u_t||_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{b}{p}||u||_p^p,$$

$$E(t) = E(u, u_t) := J(t) + \frac{1}{\rho + 2}\int_\Omega |u_t(t)|^{\rho+2}dx, \tag{2.3}$$

$$H := \{(v, w) \in \left[H_0^1(\Omega)\right]^2 : \quad I(v, w) > 0\} \cup \{(0, 0)\},$$

where

$$(g \circ v)(t) = \int_0^t g(t - \tau)||v(t) - v(\tau)||_2^2 d\tau. \tag{2.4}$$

Remark 2.3 By multiplying equation (1.1) by u_t and integrating over Ω, using integration by parts and hypothesis (G2), we get

$$E'(t) = -\left(\gamma \int_\Omega |\nabla u_t|^2 dx - \frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)||\nabla u(t)||^2\right) \leq 0,$$

for t in $[0, T)$. This means that the energy is uniformly bounded (by $E(0)$) and is decreasing in t.

Next, we would like to prove the invariance of the set H, but first let us mention here that we will be using the following Sobolev-Poincaré embedding

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega) \tag{2.5}$$

so

$$\|v\|_q \leq C\,\|\nabla v\|_2\,, \tag{2.6}$$

with $2 < q \leq 2n/(n-2)$ if $n \geq 3$ or $q > 2$ if $n = 1, 2$.

Lemma 2.2 Suppose that (G1), (G2) and the hypotheses on p and ρ hold. If $(u_0, u_1) \in H$ and satisfy

$$\beta = \frac{b}{l}C_*^p\left(\frac{2p}{(p-2)\,l}E(u_0, u_1)\right)^{(p-2)/2} < 1, \tag{2.7}$$

where C_* is the best constant in (2.6) with $q = p$, then $(u(t), u_t(t)) \in H$, for each $t \in [0, T)$.

Proof Since $I(u_0, u_1) > 0$ then, by continuity, there exists $T_m \leq T$ such that $I(u, u_t) \geq 0$ for all $t \in [0, T_m)$. This implies, for all $t \in [0, T_m)$

$$\begin{aligned} J(t) &= \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)||\nabla u(t)||_2^2 + \frac{1}{2}||\nabla u_t||_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{b}{p}||u(t)||_p^p \\ &= \frac{p-2}{2p}\left[\left(1 - \int_0^t g(s)ds\right)||\nabla u(t)||_2^2 + ||\nabla u_t||_2^2 + (g \circ \nabla u)(t)\right] + \frac{1}{p}I(u, u_t) \\ &\geq \frac{p-2}{2p}\left[\left(1 - \int_0^t g(s)ds\right)||\nabla u(t)||_2^2 + ||\nabla u_t||_2^2 + (g \circ \nabla u)(t)\right]. \end{aligned} \tag{2.8}$$

Hence, from (G1), (2.3), (2.4), (2.8) and Remark 3, we find

$$\begin{aligned} l||\nabla u(t)||_2^2 &\leq \left(1 - \int_0^t g(s)ds\right)||\nabla u(t)||_2^2 \leq \frac{2p}{p-2}J(t) \\ &\leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(u_0, u_1), \ \forall t \in [0, T_m). \end{aligned} \tag{2.9}$$

By exploiting the embedding relation (2.6) (with $q = p$ and C_* as in the statement of the lemma), (G1) and the assumption (2.7), we easily arrive at

$$\begin{aligned} b||u(t)||_p^p &\leq bC_*^p||\nabla u(t)||_2^p \leq \frac{bC_*^p}{l}||\nabla u(t)||_2^{p-2}l||\nabla u(t)||_2^2 \\ &\leq \beta l||\nabla u(t)||_2^2 \leq \beta\left(1 - \int_0^t g(s)ds\right)||\nabla u(t)||_2^2 \\ &< \left(1 - \int_0^t g(s)ds\right)||\nabla u(t)||_2^2, \ \forall t \in [0, T_m). \end{aligned} \tag{2.10}$$

Therefore,

$$I(t) = \left(1 - \int_0^t g(s)ds\right)||\nabla u||_2^2 - b||u(t)||_p^p + ||\nabla u_t||_2^2 + (g \circ \nabla u)(t) > 0, \forall t \in [0, T_m).$$

This shows that $(u(t), u_t(t) \in H$, $\forall t \in [0, T_m)$. By repeating this procedure, T_m is extended to T.

3 Exponential decay

Theorem 3.1 Suppose that (G1), (G2) and the hypotheses on p and ρ hold. If $(u_0, u_1) \in H$ and satisfy (2.7), then the solution is global in time.

Proof It suffices to show that $||\nabla u(t)||_2^2 + ||\nabla u_t||_2^2 + ||u_t(t)||_{\rho+2}^{\rho+2}$ is bounded independently of t. To achieve this note that, from (2.8), for $t \in [0, T)$

$$\begin{aligned} E(u_0, u_1) &\geq E(t) = J(t) + \frac{1}{\rho+2}||u_t(t)||_{\rho+2}^{\rho+2} \\ &\geq \frac{p-2}{2p}\left[l||\nabla u(t)||_2^2 + ||\nabla u_t(t)||_2^2 + (g \circ \nabla u)(t)\right] + \frac{1}{p}I(u, u_t) + \frac{1}{\rho+2}||u_t(t)||_{\rho+2}^{\rho+2} \\ &\geq \frac{p-2}{2p}\left[l||\nabla u(t)||_2^2 + ||\nabla u_t(t)||_2^2\right] + \frac{1}{\rho+2}||u_t(t)||_{\rho+2}^{\rho+2}, \end{aligned} \tag{3.1}$$

since $I(u,u_t)$ (by the Lemma 1) and $(g\circ\nabla u)(t)$ are positive. Therefore,

$$||\nabla u(t)||_2^2+||\nabla u_t(t)||_2^2+||u_t(t)||_{\rho+2}^{\rho+2}\leq CE(u_0,u_1),$$

where C is positive and depends only on p, ρ and l and is independent of t.

Remark 3.1 Observe that in the previous lemma and theorem we did not use hypothesis (G2). Only the non positivity of $g^{'}(t)$ was needed (see Remark 3). This will not be the case in the theorem below. Indeed, we will need $g(t)$ to decrease in an exponential fashion.

Theorem 3.2 Suppose that (G1), (G2) and the hypotheses on p and ρ hold. Assume further that $(u_0,u_1)\in H$ and verify (2.7), then there exist positive constants k and K such that the solution of (1.1) satisfies

$$E(t)\leq Ke^{-kt},\quad \forall t\geq 0.$$

Proof We define

$$F(t):=E(t)+\frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^\rho uu_t dx+\varepsilon\int_\Omega \nabla u.\nabla u_t dx+\frac{\varepsilon\gamma}{2}\int_\Omega |\nabla u|^2dx \tag{3.2}$$

for ε so small that

$$\alpha_1F(t)\leq E(t)\leq \alpha_2F(t), \tag{3.3}$$

holds for two positive constants α_1 and α_2. This is possible because of the embedding (2.5), (2.6) and the fact that $E(t)$ is uniformly bounded (see Remark 3). Indeed, we find by (3.1)

$$\begin{aligned}\int_\Omega |u_t|^\rho uu_t dx &\leq \tfrac{1}{2}\int_\Omega |u_t|^{2(\rho+1)}dx+\tfrac{1}{2}\int_\Omega |u|^2dx\\ &\leq \tfrac{C_e}{2}\left(\tfrac{2p}{p-2}\right)^\rho\int_\Omega |\nabla u_t|^2dx+\tfrac{C_p}{2}\int_\Omega |u|^2dx,\end{aligned}$$

where C_e and C_p are the embedding and the Poincaré constant, respectively. We also need the inequality

$$\int_\Omega \nabla u.\nabla u_t dx\leq\frac{1}{2}\int_\Omega|\nabla u|^2dx+\frac{1}{2}\int_\Omega|\nabla u_t|^2dx.$$

Differentiating (3.2), we obtain

$$\begin{aligned}F'(t)=&-\left(\gamma\int_\Omega|\nabla u_t|^2dx-\tfrac{1}{2}(g'\circ\nabla u)(t)+\tfrac{1}{2}g(t)||\nabla u(t)||^2\right)+\tfrac{\varepsilon}{\rho+1}\int_\Omega|u_t|^{\rho+2}dx\\ &-\varepsilon\int_\Omega|\nabla u|^2dx+\varepsilon\int_\Omega|\nabla u_t|^2dx+\varepsilon\int_0^t g(t-\tau)\int_\Omega\nabla u(t).\nabla u(\tau)dxd\tau+\varepsilon b\int_\Omega|u(x,t)|^pdx.\end{aligned} \tag{3.4}$$

Adding and substracting $\int_0^t g(t-\tau)||\nabla u(t)||_2^2d\tau$ to the right hand side of (3.5), we get

$$\begin{aligned}F'(t)\leq&-\left(\gamma\int_\Omega|\nabla u_t|^2dx-\tfrac{1}{2}(g'\circ\nabla u)(t)+\tfrac{1}{2}g(t)||\nabla u(t)||^2\right)+\tfrac{\varepsilon}{\rho+1}\int_\Omega|u_t|^{\rho+2}dx\\ &-\varepsilon\int_\Omega|\nabla u|^2dx+\varepsilon\int_\Omega|\nabla u_t|^2dx+\varepsilon b\int_\Omega|u|^pdx+\varepsilon\int_0^t g(t-\tau)||\nabla u(t)||_2^2d\tau\\ &+\varepsilon\int_0^t g(t-\tau)\int_\Omega\nabla u(t).[\nabla u(\tau)-\nabla u(t)]dxd\tau.\end{aligned} \tag{3.5}$$

We then use (2.3) and (2.10) to get for, $0 < \alpha < 1$,

$$\begin{aligned} b\int_\Omega |u(t)|^p dx &= \alpha b \int_\Omega |u(t)|^p dx + (1-\alpha) b \int_\Omega |u(t)|^p dx \\ &\le \alpha\{\tfrac{p}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx + \tfrac{p}{2}\left(1-\int_0^t g(s)ds\right)||\nabla u(t)||_2^2 + \tfrac{p}{2}||\nabla u_t||_2^2 \\ &+\tfrac{p}{2}(g\circ\nabla u)(t) - pE(t)\} + (1-\alpha)\beta\left(1-\int_0^t g(s)ds\right)||\nabla u(t)||_2^2, \end{aligned} \quad (3.6)$$

and exploit Young's inequality to derive, for any $\delta > 0$,

$$\int_0^t g(t-\tau)\int_\Omega \nabla u(t).[\nabla u(\tau)-\nabla u(t)]dxd\tau \le \frac{1}{4\delta}(g\circ\nabla u)(t)+\delta\int_0^t g(s)ds||\nabla u(t)||_2^2. \quad (3.7)$$

A combination of (3.6) - (3.8) gives

$$\begin{aligned} F'(t) \le &-\left[\gamma - \varepsilon\left(\alpha\tfrac{p}{2}+1\right)\right]\int_\Omega |\nabla u_t|^2 dx + \tfrac{1}{2}(g'\circ\nabla u)(t) - \tfrac{1}{2}g(t)||\nabla u(t)||^2 \\ &+\varepsilon\left(\tfrac{1}{\rho+1}+\tfrac{\alpha p}{\rho+2}\right)\int_\Omega |u_t|^{\rho+2}dx + \varepsilon\left(\tfrac{p\alpha}{2}+\tfrac{1}{4\delta}\right)(g\circ\nabla u)(t) \\ &+\varepsilon\left[\alpha(\tfrac{p}{2}-1)-\eta(1-\alpha)\right]\left(1-\int_0^t g(s)ds\right)||\nabla u(t)||_2^2 \\ &+\varepsilon\delta\left(\int_0^t g(s)ds\right)||\nabla u(t)||_2^2 - \varepsilon\alpha p E(t), \end{aligned}$$

where $\eta = 1-\beta$. By using (2.9) and choosing α close to 1 so that

$$\alpha\left(\frac{p}{2}-1\right) - \eta(1-\alpha) \ge 0,$$

we arrive at

$$\begin{aligned} F'(t) \le &-\left[\gamma - \varepsilon\left(\alpha\tfrac{p}{2}+1\right)\right]\int_\Omega |\nabla u_t|^2 dx + \tfrac{1}{2}(g'\circ\nabla u)(t) + \varepsilon\left(\tfrac{p\alpha}{2}+\tfrac{1}{4\delta}\right)(g\circ\nabla u)(t) \\ &+\varepsilon\left(\tfrac{1}{\rho+1}+\tfrac{\alpha p}{\rho+2}\right)\int_\Omega |u_t|^{\rho+2}dx + \varepsilon\delta\left(\int_0^t g(s)ds\right)||\nabla u(t)||_2^2 - \alpha\varepsilon p E(t) \\ &+\varepsilon[\alpha(\tfrac{p}{2}-1)-\eta(1-\alpha)]\tfrac{2p}{p-2}E(t) \end{aligned}$$

or

$$\begin{aligned} F'(t) \le &-\left[\gamma - \varepsilon\left(\alpha\tfrac{p}{2}+1\right)\right]\int_\Omega |\nabla u_t|^2 dx + \left[\tfrac{1}{2} - \varepsilon\left(\tfrac{p\alpha}{2}+\tfrac{1}{4\delta}\right)m\right](g'\circ\nabla u)(t) \\ &-\tfrac{2p\varepsilon}{p-2}\left[\eta(1-\alpha)-\delta\tfrac{1-l}{l}\right]E(t) + \varepsilon\left(\tfrac{1}{\rho+1}+\tfrac{\alpha p}{\rho+2}\right)\int_\Omega |u_t|^{\rho+2}dx. \end{aligned} \quad (3.8)$$

The last term in the right hand side of (3.11) can be estimated, in view of (3.1), as follows

$$\begin{aligned} \int_\Omega |u_t|^{\rho+2}dx &\le C_*^{\rho+2}||\nabla u_t||_2^{\rho+2} \le C_*^{\rho+2}||\nabla u_t||_2^{\rho}||\nabla u_t||_2^2 \\ &\le C_*^{\rho+2}\left(\tfrac{2p}{(p-2)}E(u_0,u_1)\right)^{\rho/2}||\nabla u_t||_2^2. \end{aligned}$$

Hence (3.11) becomes

$$F'(t) \le -\left[\gamma - \varepsilon\left(\alpha\tfrac{p}{2}+1\right) - \varepsilon C_*^{\rho+2}\left(\tfrac{1}{\rho+1}+\tfrac{\alpha p}{\rho+2}\right)\left(\tfrac{2p}{(p-2)}E(u_0,u_1)\right)^{\rho/2}\right]\int_\Omega |\nabla u_t|^2 dx$$
$$-\tfrac{2p\varepsilon}{p-2}\left[\eta(1-\alpha)-\delta\tfrac{1-l}{l}\right]E(t) + \left[\tfrac{1}{2}-\varepsilon\left(\tfrac{p\alpha}{2}+\tfrac{1}{4\delta}\right)m\right](g'\circ\nabla u)(t). \tag{3.9}$$

At this point we choose δ so small that

$$\eta(1-\alpha) - \delta\frac{1-l}{l} > 0,$$

then ε so small that, in addition to (3.3),

$$\gamma - \varepsilon\left(\alpha\frac{p}{2}+1\right) - \varepsilon C_*^{\rho+2}\left(\frac{1}{\rho+1}+\frac{\alpha p}{\rho+2}\right)\left(\frac{2p}{(p-2)}E(u_0,u_1)\right)^{\rho/2} \ge 0,$$

and

$$\frac{1}{2} - \varepsilon\left(\frac{p\alpha}{2}+\frac{1}{4\delta}\right)m \ge 0.$$

Consequently (3.12) takes the form

$$F'(t) \le -\frac{2p\varepsilon}{p-2}\left[\eta(1-\alpha)-\delta\frac{1-l}{l}\right]E(t) \le -\frac{2p\varepsilon}{(p-2)\alpha_1}\left[\eta(1-\alpha)-\delta\frac{1-l}{l}\right]F(t), \tag{3.10}$$

by virtue of (3.3). A simple integration of (3.13) then leads to

$$F(t) \le F(0)e^{-kt},$$

where

$$k = \frac{2p\varepsilon}{(p-2)\alpha_1}\left[\eta(1-\alpha)-\delta\frac{1-l}{l}\right].$$

The assertion of the theorem follows using once again (3.3). This completes the proof.

4 Exponential growth

In this section we shall prove that the energy is unbounded when the initial data are large enough in some sense. In fact, it will be proved that the L^p-norm of the solution grows unboundedly as an exponential function. This will be established here in spite of the strong damping generated by Δu_t and the strong exponential decreasingness of the relaxation function $g(t)$. Note that our previous results hold no matter how small or large are the values of γ and b. So we could have taken $\gamma = b = 1$. Let us adopt this in this section.

Theorem 4.1 Assume that $\rho \le p-2$. Then the solution of problem (1) grows up exponentially in the L^p-norm provided that

$$l > \frac{4}{(p+2)} \tag{4.1}$$

and the initial data u_0, u_1 are large enough (see (4.10) below).
Proof Suppose that $E(0)<0$ so that $E(t)\leq E(0)<0$, for all $t\geq 0$. Let us consider the functional

$$\mathcal{F}(t)=E(t)-\varepsilon\Psi(t),\ \varepsilon>0$$

where $\Psi(t)$ is defined by

$$\Psi(t):=\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx+\int_\Omega \nabla u.\nabla u_t dx,$$

and ε is to be determined later.

Firstly, observe that we can find positive real numbers a_i, $i=1,...,5$ such that $a_i=a_i(p,\rho,\varepsilon,l,|\Omega|)$ and

$$\begin{aligned}\mathcal{F}(t)\geq a_1\int_\Omega |u_t|^{\rho+2}dx+a_2\int_\Omega |\nabla u|^2 dx+\tfrac{1}{2}(g\circ\nabla u)(t)\\ +a_3\int_\Omega |\nabla u_t|^2 dx-a_4\int_\Omega |u|^p dx-a_5.\end{aligned}\tag{4.2}$$

Indeed, by Young inequality we have

$$\int_\Omega |u_t|^\rho u_t u dx\leq \frac{1}{p}\int_\Omega |u|^p dx+\frac{p-1}{p}\int_\Omega |u_t|^{\frac{(\rho+1)(p)}{p-1}}dx.$$

By our assumption $\rho\leq p-2$ we see that $\frac{p(\rho+1)}{p-1}\leq \rho+2$. Therefore,

$$\int_\Omega |u_t|^\rho u_t u dx\leq \frac{1}{p}\int_\Omega |u|^p dx+\frac{(p-1)|\Omega|}{p}+\frac{p-1}{p}\int_\Omega |u_t|^{\rho+2}dx.\tag{4.3}$$

Taking this estimate (4.3) into account, we infer that

$$\begin{aligned}\mathcal{F}(t)\geq \tfrac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx+\tfrac{1}{2}\left(1-\int_0^t g(s)ds\right)\int_\Omega |\nabla u|^2 dx+\tfrac{1}{2}\int_\Omega |\nabla u_t|^2 dx\\ +\tfrac{1}{2}(g\circ\nabla u)(t)-\tfrac{1}{p}\int_\Omega |u|^p dx-\tfrac{\varepsilon}{2}\int_\Omega |\nabla u|^2 dx-\tfrac{\varepsilon}{2}\int_\Omega |\nabla u_t|^2 dx\\ -\tfrac{1}{\rho+1}\tfrac{\varepsilon}{p}\int_\Omega |u|^p dx-\tfrac{\varepsilon(p-1)|\Omega|}{p(\rho+1)}-\tfrac{\varepsilon(p-1)}{p(\rho+1)}\int_\Omega |u_t|^{\rho+2}dx\end{aligned}$$

or

$$\begin{aligned}\mathcal{F}(t)\geq \left[\tfrac{1}{\rho+2}-\tfrac{\varepsilon(p-1)}{p(\rho+1)}\right]\int_\Omega |u_t|^{\rho+2}dx+\tfrac{1}{2}\left(1-\int_0^t g(s)ds-\varepsilon\right)\int_\Omega |\nabla u|^2 dx\\ +\tfrac{1}{2}(g\circ\nabla u)(t)+\tfrac{1}{2}(1-\varepsilon)\int_\Omega |\nabla u_t|^2 dx-\tfrac{\varepsilon(p-1)|\Omega|}{p(\rho+1)}-\tfrac{1}{p}\left(1+\tfrac{\varepsilon}{\rho+1}\right)\int_\Omega |u|^p dx.\end{aligned}$$

Choosing ε small enough (namely $\varepsilon<\min\left(l,\frac{p(\rho+1)}{(p-1)(\rho+2)}\right)$), our claim (4.2) follows.

Secondly, a differentiation of $\mathcal{F}(t)$ with respect to t yields

$$\begin{aligned}\mathcal{F}'(t)=E'(t)-\varepsilon\Psi'(t)=\tfrac{1}{2}(g'\circ\nabla u)(t)-\int_\Omega |\nabla u_t|^2 dx-\tfrac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx\\ -\varepsilon\left\{-\int_\Omega |\nabla u|^2 dx+\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx+\int_\Omega |\nabla u_t|^2 dx\right.\\ \left.+\tfrac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}dx-\int_\Omega \nabla u\nabla u_t dx+\int_\Omega |u|^p dx\right\}.\end{aligned}\tag{4.4}$$

We would like to estimate the fifth term in the right hand side of this identity (4.4). By (G1) we have

$$\begin{array}{c} \int\limits_\Omega \nabla u(t) \int\limits_0^t g(t-s)\nabla u(s)dsdx \\ \leq \int\limits_\Omega |\nabla u(t)| \int\limits_0^t g(t-s)\left(|\nabla u(t)-\nabla u(s)|+|\nabla u(t)|\right)dsdx \\ \leq (1-l)\int\limits_\Omega |\nabla u|^2\,dx + \int\limits_\Omega |\nabla u(t)| \int\limits_0^t g(t-s)\,|\nabla u(t)-\nabla u(s)|\,dsdx. \end{array}$$

Using Young inequality we obtain

$$\begin{array}{c} \int\limits_\Omega \nabla u(t) \int\limits_0^t g(t-s)\nabla u(s)dsdx \\ \leq (1-l+\delta_1)\int\limits_\Omega |\nabla u|^2\,dx + \frac{1-l}{4\delta_1}(g\circ\nabla u)(t),\ \ \delta_1>0. \end{array} \tag{4.5}$$

We shall also use the inequality

$$\int\limits_\Omega \nabla u \nabla u_t dx \leq \delta_2 \int\limits_\Omega |\nabla u|^2\,dx + \frac{1}{4\delta_2}\int\limits_\Omega |\nabla u_t|^2\,dx,\ \ \delta_2>0. \tag{4.6}$$

Let us add and substract $k\varepsilon\mathcal{F}(t)$ to the right hand side of (4.4), then from (4.5) it appears that

$$\begin{array}{l} \mathcal{F}'(t) \leq k\varepsilon\mathcal{F}(t) - k\varepsilon\left\{\frac{1}{\rho+2}\int\limits_\Omega |u_t|^{\rho+2}\,dx + \frac{1}{2}\left(1-\int\limits_0^t g(s)ds\right)\int\limits_\Omega |\nabla u|^2\,dx\right. \\ +\frac{1}{2}\int\limits_\Omega |\nabla u_t|^2\,dx + \frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{p}\int\limits_\Omega |u|^p\,dx - \frac{\varepsilon}{\rho+1}\int\limits_\Omega |u_t|^\rho\, u_t u dx \\ \left.+\varepsilon\int\limits_\Omega u\Delta u_t dx\right\} + \frac{1}{2}(g'\circ\nabla u)(t) - \int\limits_\Omega |\nabla u_t|^2\,dx - \frac{1}{2}g(t)\int\limits_\Omega |\nabla u|^2\,dx \\ +\varepsilon\int\limits_\Omega |\nabla u|^2\,dx - \varepsilon\int\limits_\Omega |\nabla u_t|^2\,dx - \frac{\varepsilon}{\rho+1}\int\limits_\Omega |u_t|^{\rho+2}\,dx + \varepsilon\int\limits_\Omega \nabla u\nabla u_t dx \\ -\varepsilon\int\limits_\Omega |u|^p\,dx + \varepsilon(1-l+\delta_1)\int\limits_\Omega |\nabla u|^2\,dx + \frac{\varepsilon(1-l)}{4\delta_1}(g\circ\nabla u)(t). \end{array}$$

Next, using (4.3), (4.6) and the assumption $g(t) \leq -mg'(t)$, $t \geq 0$ (see (G2)), we entail

$$\begin{array}{c} \mathcal{F}'(t) \leq k\varepsilon\mathcal{F}(t) - \varepsilon\left[\frac{k}{\rho+2} + \frac{1}{\rho+1} - \frac{k\varepsilon(p-1)}{p(\rho+1)}\right]\int\limits_\Omega |u_t|^{\rho+2}\,dx \\ -\varepsilon\left[\frac{k+2}{2}l - 2 - \delta_1 - (k\varepsilon+1)\delta_2\right]\int\limits_\Omega |\nabla u|^2\,dx - \left[1+\frac{(k+2)\varepsilon}{2} - \frac{\varepsilon(k\varepsilon+1)}{4\delta_2}\right]\int\limits_\Omega |\nabla u_t|^2\,dx \\ -\varepsilon\left[1-\frac{k}{p} - \frac{k\varepsilon}{p(\rho+1)}\right]\int\limits_\Omega |u|^p\,dx - \frac{1}{2}\left[\frac{1}{m} + k\varepsilon - \frac{\varepsilon(1-l)}{2\delta_1}\right](g\circ\nabla u)(t) + \frac{k\varepsilon^2(p-1)|\Omega|}{p(\rho+1)}. \end{array} \tag{4.7}$$

Choosing k, $0<\varepsilon<1$, δ_1 and δ_2 appropriately we can make all the terms in brackets in (4.7) nonnegative. Indeed, observe that the first term in brackets is nonnegative as soon as

$$\varepsilon \leq \varepsilon_1 := \frac{p}{k(p-1)}\left(1+\frac{k(\rho+1)}{\rho+2}\right).$$

Since $p > \frac{2(2-l)}{l}$ (this is guaranteed by (4.1)), we may pick k such that

$$p > k > \frac{2(2-l)}{l}.$$

Then, the fourth term in brackets is nonnegative provided that

$$\varepsilon \leq \varepsilon_2 := \frac{(p-k)(\rho+1)}{k},$$

and the second term is nonnegative if we select δ_1 and δ_2 so small that

$$\delta_1 + (k+1)\delta_2 \leq \frac{k+2}{2}l - 2.$$

As for the third and fifth term we pick

$$\varepsilon \leq \varepsilon_3 := 4\delta_2 / \left[k + 1 - 2\delta_2(k+2)\right]$$

and

$$\varepsilon \leq \varepsilon_4 := 2\delta_1 / m\left[1 - l - 2k\delta_1\right],$$

respectively.

Finally, choosing

$$\varepsilon \leq \min\left\{\varepsilon_i : i = 1, 2, 3, 4\right\},$$

we obtain from (4.7),

$$\mathcal{F}^{'}(t) \leq k\varepsilon\mathcal{F}(t) + \Lambda, \;\; t > 0 \tag{4.8}$$

where

$$\Lambda = \frac{k\varepsilon^2(p-1)\left|\Omega\right|}{p(\rho+1)}.$$

An integration of (4.8) gives for $M(t) = -\mathcal{F}(t)$,

$$M(t) \geq \left(M(0) - \frac{\Lambda}{k\varepsilon}\right)e^{k\varepsilon t} + \frac{\Lambda}{k\varepsilon} \geq \left(M(0) - \frac{\Lambda}{k\varepsilon}\right)e^{k\varepsilon t}. \tag{4.9}$$

The initial data are assumed to satisfy $M(0) - \frac{\Lambda}{k\varepsilon} =: b > 0$ or

$$-\mathcal{F}(0) - \frac{\varepsilon(p-1)\left|\Omega\right|}{p(\rho+1)} > 0,$$

that is,

$$\begin{aligned} &\tfrac{1}{p}\int_\Omega |u_0|^p\,dx - \tfrac{1}{\rho+2}\int_\Omega |u_1|^{\rho+2}\,dx - \tfrac{1}{2}\int_\Omega |\nabla u_0|^2\,dx - \tfrac{1}{2}\int_\Omega |\nabla u_1|^2\,dx \\ &+\tfrac{\varepsilon}{\rho+1}\int_\Omega |u_1|^\rho\, u_1 u_0 dx + \varepsilon\int_\Omega \nabla u_0 \nabla u_1 dx - a_5 > 0. \end{aligned} \tag{4.10}$$

The relations (4.2) and (4.9) imply

$$a_4\int_\Omega |u|^p\,dx + a_5 \geq -\mathcal{F}(t) = M(t) \geq be^{k\varepsilon t}.$$

The proof is now complete.

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