Blow up in a nonlinearly damped wave equation

Salim A. Messaoudi
Mathematical Sciences Department
KFUPM, Dhahran 31261
Saudi Arabia.
Email : messaoud@kfupm.edu.sa

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Abstract

In this paper we consider the nonlinearly damped semilinear wave equation
$$u_{tt} - \Delta u + a u_t |u_t|^{m-2} = b u |u|^{p-2}$$
associated with initial and Dirichlet boundary conditions. We prove that any strong solution, with negative initial energy, blows up in finite time if $p > m$. This result improves an earlier one in [2].
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1 Introduction

In this paper we are concerned with the following initial boundary value problem
.
$$\begin{aligned} u_{tt} - \Delta u + a u_t |u_t|^{m-2} &= b u |u|^{p-2}, \quad x \in \Omega, \quad t > 0 \\ u(x,t) &= 0, \quad x \in \partial\Omega, \ t \geq 0 \\ u(x,0) = u_0(x), &\quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{aligned} \tag{1.1}$$

where $a, b > 0$, $p, m > 2$, and Ω is a a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. For $b = 0$, it is well known that the damping term $a u_t |u_t|^{m-2}$ assures global existence for arbitrary initial data (see [3], [5]). If $a = 0$ then the source term $b u |u|^{p-2}$ causes finite time blow up of solutions with negative initial energy (see [1], [4], [6], [7]).

The interaction between the damping and the source terms was first considered by Levine [6], [7] in the linear damping case $(m = 2)$. He showed that solutions with negative initial energy blow up in finite time. Recently Georgiev and Todorova [2] extended Levine's result to the nonlinear case $(m > 2)$. In their work, the authors introduced a different method and determined suitable relations between m and p, for which there is global existence or alternatively finite time blow up. Precisely;

they showed that solutions with negative energy continue to exist globally 'in time' if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative.

This result has been lately generalized to an abstract setting and to unbounded domains by Levine and Serrin [8] and Levine, Park, and Serrin [9]. In these papers, the authors showed that no solution with negative energy can be extended on $[0, \infty)$ if $p > m$ and proved several noncontinuation theorems. This generalization allowed them also to apply their result to quasilinear situations, of which problem (1.1) is a particular case.

Vitillaro [10] combined the arguments in [2] and [8] to extend these results to situations where the damping is nonlinear and the solution has positive initial energy.

In this work, we prove the same result of [2] without imposing the condition that the initial energy is sufficiently negative. In other words, we show that any solution of (1.1) with negative initial energy - however close to zero is - blows up in finite time. In addition to ommitting the condition of large 'negative' initial data, our technique of proof is simpler than the ones in [2] and [8]. We first state a local result established in [2].

Theorem 1.1. *Suppose that* $m > 2$, $p > 2$, *and*

$$p \leq 2\frac{n-1}{n-2}, \quad n \geq 3. \tag{1.2}$$

Assume further that

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \tag{1.3}$$

Then the problem (1.1) has a unique local solution

$$u \in C\left([0,T); H_0^1(\Omega)\right), \quad u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^m(\Omega \times (0,T)), \tag{1.4}$$

T is small.

Remark 1.1 The condition on p, in (1.2), is needed to establish the local existence result (see [2]). In fact under this condition, the nonlinearity is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$.

2 Main Result.

In this section we show that the solution (1.4) blows up in finite time if $p > m$ and $E(0) < 0$, where

$$E(t) := \frac{1}{2}\int_\Omega [u_t^2 + |\nabla u|^2](x,t)dx - \frac{b}{p}\int_\Omega |u(x,t)|^p dx. \tag{2.1}$$

Lemma 2.1. *Suppose that (1.2) holds. Then there exists a positive constant $C > 1$ depending on Ω only such that*

$$\|u\|_p^s \leq C\left(\|\nabla u\|_2^2 + \|u\|_p^p\right) \tag{2.2}$$

for any $u \in H_0^1(\Omega)$ and $2 \le s \le p.$

Proof. If $\|u\|_p \le 1$ then $\|u\|_p^s \le \|u\|_p^2 \le C\|\nabla u\|_2^2$ by Sobolev embedding theorems. If $\|u\|_p > 1$ then $\|u\|_p^s \le \|u\|_p^p.$ Therefore (2.2) follows.

We set

$$H(t) := -E(t)$$

and use, throughout this paper, C to denote a generic positive constant depending on Ω only. As a result of (2.1) - (2.3), we have

Corollary 2.2. Let the assumptions of the lemma hold. Then we have

$$\|u\|_p^s \le C\left(|H(t)| + \|u_t\|_2^2 + \|u\|_p^p\right) \tag{2.3}$$

for any $u \in H_0^1(\Omega)$ and $2 \le s \le p.$

Theorem 2.3. Let the conditions of the theorem 1.1 be fulfilled. Assume further that $p > m$ and

$$E(0) < 0. \tag{2.4}$$

Then the solution (1.4) blows up in finite time.

Remark 2.1. Note that contrary to [2], no condition on the size of the initial data has been done. The blow up takes place for any initial data satisfying (2.4).

Proof.

We multiply equation (1.1) by u_t and integrate over Ω to get

$$E'(t) = -a\int_\Omega |u_t(x,t)|^m dx, \tag{2.5}$$

for almost every t in $[0,T)$ since $E'(t)$ is absolutely continuous (see [2]); hence $H'(t) \ge 0.$ So we have

$$0 < H(0) \le H(t) \le \frac{b}{p}\|u\|_p^p, \tag{2.6}$$

for every t in $[0,T)$, by virtue of (2.4). We then define

$$L(t) := H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t(x,t)dx \tag{2.7}$$

for ε small to be chosen later and

$$0 < \alpha \le \min\left\{\frac{(p-2)}{2p}, \frac{(p-m)}{p(m-1)}\right\}. \tag{2.8}$$

By taking a derivative of (2.7) and using equation (1.1) we obtain

$$L'(t) := (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega [u_t^2 - |\nabla u|^2](x,t)dx \tag{2.9}$$

$$+\varepsilon b\int_\Omega |u(x,t)|^p dx - a\varepsilon\int_\Omega |u_t|^{m-2}u_t u(x,t)dx.$$

We then exploit Young's inequality

$$XY \le \frac{\delta^r}{r}X^r + \frac{\delta^{-q}}{q}Y^q; \quad X, Y \ge 0, \ \forall\delta > 0, \quad \frac{1}{r} + \frac{1}{q} = 1$$

with $r = m$ and $q = m/(m-1)$ to estimate the last term in (2.9) as follows

$$\int_\Omega |u_t|^{m-1}|u|dx \le \frac{\delta^m}{m}\|u\|_m^m + \frac{m-1}{m}\delta^{-m/(m-1)}\|u_t\|_m^m$$

which yields, by substitution in (2.9),

$$L'(t) \ge \left[(1-\alpha)H^{-\alpha}(t) - \frac{m-1}{m}\varepsilon\delta^{-m/(m-1)}\right]H'(t) + \varepsilon\int_\Omega [u_t^2 - |\nabla u|^2](x,t)dx$$
$$+\varepsilon\left[pH(t) + \frac{p}{2}\int_\Omega [u_t^2 + |\nabla u|^2](x,t)dx\right] - \varepsilon a\frac{\delta^m}{m}\|u\|_m^m; \quad \forall\delta > 0: \tag{2.10}$$

Of course (2.10) remains valid even if δ is time dependant since the integral is taken over the x variable. Therefore by taking δ so that $\delta^{-m/(m-1)} = kH^{-\alpha}(t)$, for large k to be specified later, and substituting in (2.10) we arrive at

$$L'(t) \ge \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)dx \tag{2.11}$$
$$+\varepsilon\left(\frac{p}{2}-1\right)\int_\Omega |\nabla u|^2(x,t)dx + \varepsilon\left[pH(t) - \frac{k^{1-m}}{m}aH^{\alpha(m-1)}(t)\|u\|_m^m\right]:$$

By exploiting (2.6) and the inequality $\|u\|_m^m \le C\|u\|_p^m$, we obtain

$$H^{\alpha(m-1)}(t)\|u\|_m^m \le \left(\frac{b}{p}\right)^{\alpha(m-1)} C\|u\|_p^{m+\alpha p(m-1)};$$

hence (2.11) yields

$$L'(t) \ge \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)dx \tag{2.12}$$
$$+\varepsilon\left(\frac{p}{2}-1\right)\int_\Omega |\nabla u|^2(x,t)dx + \varepsilon\left[pH(t) - \frac{k^{1-m}}{m}a\left(\frac{b}{p}\right)^{\alpha(m-1)} C\|u\|_p^{m+\alpha p(m-1)}\right]:$$

We then use corollary 2.2 and (2.8), for $s = m + \alpha p(m-1) \le p$; to deduce from (2.12)

$$L'(t) \ge \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)dx \tag{2.13}$$
$$+\varepsilon\left(\frac{p}{2}-1\right)\int_\Omega |\nabla u|^2(x,t)dx + \varepsilon\left[pH(t) - C_1k^{1-m}\left\{H(t) + \|u_t\|_2^2 + \|u\|_p^p\right\}\right];$$

where $C_1 = a\left(\frac{b}{p}\right)^{\alpha(m-1)}C/m$: By noting that

$$H(t) = \frac{b}{p}\|u\|_p^p - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2$$

and writing $p = (p+2)/2 + (p-2)/2$, (2.13) yields

$$L'(t) \geq \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\frac{p-2}{4}\|\nabla u\|_2^2 \tag{2.14}$$

$$+\varepsilon\left[\left(\frac{p+2}{2} - C_1 k^{1-m}\right)H(t) + \left(\frac{p-2}{2p}b - C_1k^{1-m}\right)\|u\|_p^p + \left(\frac{p+6}{4} - C_1 k^{1-m}\right)\|u_t\|_2^2\right]$$

At this point, we choose k large enough so that the coefficients of $H(t); \|u_t\|_2^2$; and $\|u\|_p^p$ in (2.14) are strictly positive; hence we get

$$L'(t) \geq \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\gamma\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p\right]; \tag{2.15}$$

where $\gamma > 0$ is the minimum of these coefficients. Once k is fixed (hence γ), we pick ε small enough so that $(1-\alpha) - \varepsilon k(m-1)/m \geq 0$ and

$$L(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0 u_1(x)dx > 0.$$

Therefore (2.15) takes the form

$$L'(t) \geq \gamma\varepsilon\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p\right]. \tag{2.16}$$

Consequently we have

$$L(t) \geq L(0) > 0; \qquad \forall\, t \geq 0.$$

Next we would like to show that

$$L'(t) \geq \Gamma L^{1/(1-\alpha)}(t); \forall t \geq 0; \tag{2.17}$$

where Γ is a positive constant depending on $\varepsilon\gamma$ and C (the constant of lemma 2.1). Once (2.17) is established, we obtain in a standard way the finite time blow up of $L(t)$; hence of u (see [1] for instance).

To prove (2.17), we first estime

$$\left|\int_\Omega uu_t(x;t)dx\right| \leq \|u\|_2\|u_t\|_2 \leq C\|u\|_p\|u_t\|_2$$

which implies

$$\left|\int_\Omega uu_t(x;t)dx\right|^{1/(1-\alpha)} \leq C\|u\|_p^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}.$$

Again Young's inequality gives us

$$\left|\int_\Omega uu_t(x;t)dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_p^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\right]; \tag{2.18}$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\alpha)$, to get $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p$ by (2.8). Therefore (2.18) becomes

$$\left|\int_\Omega uu_t(x;t)dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_p^s + \|u_t\|_2^2\right];$$

where $s = 2/(1-2\alpha) \le p$. By using corollary 2.2 we obtain

$$\left|\int_\Omega u u_t(x,t)dx\right|^{1/(1-\alpha)} \le C\left[H(t) + \|u\|_p^p + \|u_t\|_2^2\right], \qquad \forall t \ge 0. \tag{2.19}$$

Finally by noting that

$$L^{1/(1-\alpha)}(t) = \left(H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t(x,t)dx\right)^{1/(1-\alpha)}$$
$$\le 2^{1/(1-\alpha)}\left(H(t) + \left|\int_\Omega u u_t(x,t)dx\right|^{1/(1-\alpha)}\right)$$

and combining it with (2.16) and (2.19), the inequality (2.17) is established. This completes the proof.

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