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A NONLOCAL MIXED SEMILINEAR PROBLEM FOR SECOND-ORDER HYPERBOLIC EQUATIONS

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Abstract. In this work we study a nonlinear hyperbolic one-dimensional problem with a nonlocal condition. We establish a blow up result for large initial data and a decay result for small initial data.

1. Introduction

In the region $Q = (0, a) \times (0, T)$, with $a < \infty$, $T < \infty$, we consider the following one-dimensional semilinear hyperbolic nonlocal problem

$$\begin{gathered} u_{tt} + u_t - \frac{1}{x}\left(xu_x\right)_x = |u|^{p-2}u \\ u(a,t) = 0, \quad \int_0^a xu(x,t)dx = 0 \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \end{gathered} \tag{1.1}$$

for $p > 2$. The mathematical modelling by evolution problems with a nonlocal constraint of the form $\int_0^a \xi(x)u(x,t)dx = \gamma(t)$ is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physics. See for example Cahlon [2], Cannon [3], Ionkin [8], Kamynin [9], Shi and Shilor [16], Choi and chan [4], Samarskii [15], and Ewing [5]. The first paper that discussed second-order partial differential equations with nonlocal integral conditions goes back to Cannon *et al* [3]. In fact most of the research by then was devoted to the classical solutions (see [3] and the references therein for more information regarding this matter). Later, mixed problems with integral conditions for both parabolic and hyperbolic equations were studied by Gordeziani and Avalishvili [7], Ionkin [8], Kamynin [9], Mesloub and Bouziani [10, 11], Mesloub and Messaoudi [12], Pulkina [13,14], Volkodavov and Zhukov [17], and Yurchuk [18]. We should mention here that the presence of the integral term in the boundary condition can greatly complicate the application of standard functional techniques.

This paper is organized as follows: In section 2, we state the related linear problem, introduce appropriate function spaces to be used and present an abstract formulation of the posed linear problem. In section 3, we establish a priori bound,

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from which we deduce the uniqueness and continuous dependence of a solution on the data. Section 4 is devoted to the solvability of the linear problem. In section 5, we state and prove the local existence result for the semilinear problem (1.1). In section 6, we show that the solution of (1.1) blows up in finite time if the initial energy is negative. Finally, in section 7 we show that the solution of (1.1) decays exponentially for positive but sufficiently small initial energy.

2. The linear Problem

In this section we study a linear problem related to (1.1) and establish a strong solution. Thus we consider

$$\mathcal{L}u = u_{tt} + u_t - \frac{1}{x}\left(xu_x\right)_x = f(x,t), \tag{2.1}$$

$$\ell_1 u = u(x,0) = \varphi_1(x), \tag{2.2}$$

$$\ell_2 u = u_t(x,0) = \varphi_2(x), \tag{2.3}$$

$$u(a,t) = 0, \tag{2.4}$$

$$\int_0^a xu(x,t)dx = 0. \tag{2.5}$$

To study our problem, we introduce appropriate function spaces. Let $L^2_\rho(Q)$ be the weighted L^2-space with the norm

$$\|u\|^2_{L^2_\rho(Q)} = \int_Q xu^2 dxdt$$

and the scalar product $(u,v)_{L^2_\rho(Q)} = (xu,v)_{L^2(Q)}$. Let $V^{1,0}_\rho(Q)$ and $V^{1,1}_\rho(Q)$ be the Hilbert spaces with scalar products respectively

$$(u,v)_{V^{1,0}_\rho(Q)} = (u,v)_{L^2_\rho(Q)} + (u_x,v_x)_{L^2_\rho(Q)},$$
$$(u,v)_{V^{1,1}_\rho(Q)} = (u,v)_{L^2_\rho(Q)} + (u_x,v_x)_{L^2_\rho(Q)} + (u_t,v_t)_{L^2_\rho(Q)},$$

and with associated norms:

$$\|u\|^2_{V^{1,0}_\rho(Q)} = \|u\|^2_{L^2_\rho(Q)} + \|u_x\|^2_{L^2_\rho(Q)},$$
$$\|u\|^2_{V^{1,1}_\rho(Q)} = \|u\|^2_{L^2_\rho(Q)} + \|u_x\|^2_{L^2_\rho(Q)} + \|u_t\|^2_{L^2_\rho(Q)}.$$

The problem (2.1)-(2.5) can be considered as solving the operator equation

$$Lu = (\mathcal{L}u, \ell_1 u, \ell_2 u) = (f, \varphi_1, \varphi_2) = \mathcal{F},$$

where L is an operator defined on E into F. E is the Banach space of functions $u \in L^2_\rho(Q)$, satisfying conditions (2.4) and (2.5) with the norm

$$\|u\|^2_E = \sup_{0\le\tau\le T} \|u(.,\tau\|^2_{V^{1,1}_\rho((0,a))}$$

and F is the Hilbert space $L^2_\rho(Q)\times V^{1,0}_\rho(0,a) \times L^2_\rho(0,a)$ which consists of elements $\mathcal{F} = (f, \varphi_1, \varphi_2)$ with the norm

$$\|\mathcal{F}\|^2_F = \|\varphi_1\|^2_{V^{1,0}_\rho((0,a))} + \|\varphi_2\|^2_{L^2_\rho((0,a))} + \|f\|^2_{L^2_\rho(Q)}.$$

Let $D(L)$ be the set of all functions $u \in L^2(Q)$, for which $u_t, u_{tt}, u_x, u_{xx}, u_{xt} \in L^2(Q)$ and satisfying conditions (2.4) and (2.5).

3. A priori bound

Theorem 3.1. *There exists a positive constant c, such that for each function $u \in D(L)$ we have*

$$\|u\|_E \le c\|Lu\|_F. \tag{3.1}$$

Proof. Taking the scalar product in $L^2(Q^\tau)$ of equation (2.1) and the integro-differential operator

$$\mathcal{M}u = -x(\tau - t)\int_0^t (\Im_x(\xi u_t))(\xi, s)ds + xu_t,$$

where $Q^\tau = (0,a)\times(0,\tau)$ and $\Im_x(\zeta v) = \int_0^x \zeta v(\zeta,t)d\zeta$, we obtain

$$\begin{aligned}
&- \big((\tau - t)u_{tt}, \int_0^t (\Im_x(\xi u_t))(x,s)ds\big)_{L^2_\rho(Q^\tau)} \\
&+ \big((\tau - t)(xu_x)_x, \int_0^t (\Im_x(\xi u_t))(x,s)ds\big)_{L^2(Q^\tau)} \\
&+ (u_{tt}, u_t)_{L^2_\rho(Q^\tau)} - \big(u_t, (xu_x)_x\big)_{L^2(Q^\tau)} + \|u_t\|^2_{L^2_\rho(Q^\tau)} \\
&- \big((\tau - t)u_t, \int_0^t (\Im_x(\xi u_t))(x,s)ds\big)_{L^2_\rho(Q^\tau)} \\
&= (\mathcal{L}u, \mathcal{M}u)_{L^2(Q^\tau)}.
\end{aligned} \tag{3.2}$$

Successive integration by parts of integrals on the left-hand side of (3.2) are straightforward but somewhat tedious. We give only their results

$$-((\tau - t)u_{tt}, \int_0^t (\Im_x(\xi u_t))(x,s)ds)_{L^2_\rho(Q^\tau)} = -(\int_0^t \Im_x(\xi u_t)ds, u_t)_{L^2_\rho(Q^\tau)}\ , \tag{3.3}$$

$$\begin{aligned}
&((\tau - t)(xu_x)_x, \int_0^t (\Im_x(\xi u_t))(x,s)ds)_{L^2(Q^\tau)} \\
&= -(x(\tau - t)u_x, u)_{L^2_\rho(Q^\tau)} + (x(\tau - t)u_x, \varphi)_{L^2_\rho(Q^\tau)},
\end{aligned} \tag{3.4}$$

$$(u_{tt}, u_t)_{L^2_\rho(Q^\tau)} = \frac{1}{2}\|u_t(x.\tau)\|^2_{L^2_\rho(0,a)} - \frac{1}{2}\|\varphi_2\|^2_{L^2_\rho(0,a)}, \tag{3.5}$$

$$-(u_t, (xu_x)_x)_{L^2(Q^\tau)} = \frac{1}{2}\|u_x(x.\tau)\|^2_{L^2_\rho(0,a)} - \frac{1}{2}\|\partial\varphi_1/\partial x\|^2_{L^2_\rho(0,a)}.$$

By substituting (3.3)-(3.5) in (3.2), we obtain

$$\begin{aligned}
&\|u_t(x.\tau)\|^2_{L^2_\rho((0,a))} + \|u_x(x.\tau)\|^2_{L^2_\rho((0,a))} + 2\|u_t\|^2_{L^2_\rho(Q^\tau)} \\
&= \|\varphi_2\|^2_{L^2_\rho((0,a))} + \|\partial\varphi_1/\partial x\|^2_{L^2_\rho((0,a))} \\
&\quad + 2(x(\tau - t)u_x, u)_{L^2_\rho(Q^\tau)} - 2(x(\tau - t)u_x, \varphi_1)_{L^2_\rho(Q^\tau)}, \\
&\quad + 2((\tau - t)u_t, \int_0^t (\Im_x(\xi u_t))(x,s)ds)_{L^2_\rho(Q^\tau)} \\
&\quad - 2((\tau - t)\int_0^t (\Im_x(\xi u_t))(\xi,s)ds, \mathcal{L}u)_{L^2_\rho(Q^\tau)} \\
&\quad + 2(u_t, \mathcal{L}u)_{L^2_\rho(Q^\tau)} + 2(u_t, \int_0^t (\Im_x(\xi u_t))(x,s)ds)_{L^2_\rho(Q^\tau)}.
\end{aligned} \tag{3.6}$$

Estimates for the last six terms on the right-hand side of (3.6) are as follows:

$$2(x(\tau-t)u_x,u)_{L^2_\rho(Q^\tau)} \le Ta\|u_x\|^2_{L^2_\rho(Q^\tau)} + Ta\|u\|^2_{L^2_\rho(Q^\tau)}, \tag{3.7}$$

$$-2(x(\tau-t)u_x,\varphi_1) \le Ta\|u_x\|^2_{L^2_\rho(Q^\tau)} + T^2a\|\varphi_1\|^2_{L^2_\rho((0,a))}, \tag{3.8}$$

$$\begin{aligned} &2((\tau-t)u_t,\int_0^t(\Im_x(\xi u_t))(x,s)ds)_{L^2_\rho(Q^\tau)}\\ &= 2((\tau-t)u_t,\Im_x(\xi u))_{L^2_\rho(Q^\tau)} - 2((\tau-t)u_t,\Im_x(\xi\varphi_1))_{L^2_\rho(Q^\tau)}\\ &\le 2aT\|u_t\|^2_{L^2_\rho(Q^\tau)} + \frac{Ta^3}{2}\|u\|^2_{L^2_\rho(Q^\tau)} + \frac{T^2a^3}{2}\|\varphi_1\|^2_{L^2_\rho((0,a))}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &-2((\tau-t)\int_0^t(\Im_x(\xi u_t))(\xi,s)ds,\mathcal{L}u)_{L^2_\rho(Q^\tau)}\\ &= -2((\tau-t)\mathcal{L}u,\Im_x(\xi u))_{L^2_\rho(Q^\tau)} + 2((\tau-t)\mathcal{L}u,\Im_x(\xi\varphi_1))_{L^2_\rho(Q^\tau)}\\ &\le 2Ta\|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)} + \frac{Ta^3}{2}\|u\|^2_{L^2_\rho(Q^\tau)} + \frac{T^2a^3}{2}\|\varphi_1\|^2_{L^2_\rho((0,a))}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &2(u_t,\int_0^t(\Im_x(\xi u_t))(x,s)ds)_{L^2_\rho(Q^\tau)}\\ &= 2(u_t,\Im_x(\xi u))_{L^2_\rho(Q^\tau)} - 2(u_t,\Im_x(\xi\varphi_1))_{L^2_\rho(Q^\tau)}\\ &\le 2a\|u_t\|^2_{L^2_\rho(Q^\tau)} + \frac{a^3}{2}\|u\|^2_{L^2_\rho(Q^\tau)} + \frac{Ta^3}{2}\|\varphi_1\|^2_{L^2_\rho((0,a))}, \end{aligned} \tag{3.11}$$

$$2(u_t,\mathcal{L}u)_{L^2_\rho(Q^\tau)} \le \|u_t\|^2_{L^2_\rho(Q^\tau)} + \|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)}, \tag{3.12}$$

thanks to Young's inequality and to the inequality of poincare type

$$\|\Im_x(\xi u_t)\|^2_{L^2(Q)} \le \frac{a^3}{2}\|u_t\|^2_{L^2_\rho(Q)}. \tag{3.13}$$

We also have, by straight forward calculations,

$$\|u(.,\tau)\|^2_{L^2_\rho(0,a)} \le \|u\|^2_{L^2_\rho(Q^\tau)} + \|u_t\|^2_{L^2_\rho(Q^\tau)} + \|\varphi_1\|^2_{L^2_\rho((0,a))}. \tag{3.14}$$

The combination of (3.6)-(3.12) and (3.14) yields

$$\|u(.,\tau)\|^2_{V^{1,1}_\rho((0,a))} \le k\Big\{\|u\|^2_{V^{1,1}_\rho(Q^\tau)} + \|\varphi_1\|^2_{V^{1,0}_\rho(0,a)} + \|\varphi_2\|^2_{L^2_\rho(0,a)} + \|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)}\Big\}, \tag{3.15}$$

where

$$k = \max\Big\{1 + Ta^3 + \frac{a^3}{2} + Ta,\ T^2a + \frac{3T^2a^3}{2} + 1,\ 2aT + 2a,\ 2aT + 1\Big\}.$$

□

Lemma 3.2. *Let $f(t), g(t)$ and $h(t)$ be nonnegative functions on the interval $[0,T]$, such that $f(t)$ and $g(t)$ are integrable and $h(t)$ is nondecreasing. Then*

$$\int_0^\tau f(t)dt + g(\tau) \le h(\tau) + m\int_0^\tau g(t)dt$$

implies

$$\int_0^\tau f(t)dt + g(\tau) \le e^{m\tau}h(\tau).$$

The proof of this lemma is similar to lemma 7.1 in [6].

Now, applying the above lemma to the estimate (3.15), we obtain

$$\|u(.,\tau)\|^2_{V^{1,1}_\rho(0,a)} \le ke^{kT}\big\{\|\varphi_1\|^2_{V^{1,0}_\rho(0,a)} + \|\varphi_2\|^2_{L^2_\rho(0,a)} + \|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)}\big\}. \tag{3.16}$$

The right-hand side of (3.16) is independent of τ. By taking the least upper bound of the left side with respect to τ from 0 to T, we get the desired estimate (3.1) with $c = k^{1/2}e^{kT/2}$. It can be proved in a standard way that the operator L is closable (see, e.g., [10]).

Definition Let $\overline{L}$ be the closure of the operator L with domain of definition $D(\overline{L})$. A solution of the operator equation $\overline{L}u = \mathcal{F}$ is called a *strong solution* of problem (2.1)-(2.5).

By passing to the limit, the estimate (3.1) can be extended to strong solutions, that is we have the inequality

$$\|u\|_E \le c\|\overline{L}u\|_F \qquad \forall u \in D(\bar{\overline{L}}).$$

From this inequality, we deduce the following statements.

Corollary 3.3. *If a strong solution of (2.1)-(2.5) exists, it is unique and depends continuously on the elements $\mathcal{F} = (f, \varphi_1, \varphi_2) \in F$.*

Corollary 3.4. *The range $R(\overline{L})$ of the operator $\overline{L}$ is closed in F and $R(\overline{L}) = \overline{R(L)}$.*

Hence, to prove that a strong solution of problem (2.1)-(2.5) exists for any element $(f, \varphi_1, \varphi_2) \in F$, it remains to prove that $\overline{R(L)} = F$.

4. Solvability of the linear problem

To prove that the range of L is dense in F, we need first to prove the following theorem.

Theorem 4.1. *If for some function $\Psi \in L^2(Q)$ and all $u \in D(L)$, such that $\ell_1 u = \ell_2 u = 0$, we have*

$$(\mathcal{L}u, \Psi)_{L^2_\rho(Q)} = 0, \tag{4.1}$$

then Ψ vanishes almost everywhere in the domain Q.

Note that (4.1) holds for any function in $D(L)$ such that $\ell_1 u = \ell_2 u = 0$, so it can be expressed in a particular form. We consider the equation

$$u_{tt} = h(x,t) - \Im_x(\xi u_t) + u \tag{4.2}$$

where

$$h(x,t) = \int_t^T \Psi(x,s)ds \tag{4.3}$$

and

$$u(x,t) = \begin{cases} 0 & 0 \le t \le s \\ \int_s^t (t-\tau)\cdot u_{\tau\tau}d\tau & s \le t \le T\,. \end{cases} \tag{4.4}$$

It follows from (4.2)-(4.4) that

$$\Psi = -u_{ttt} - \Im_x(\xi u_t) + u_t \tag{4.5}$$

Lemma 4.2. *The function Ψ defined above is in $L^2_\rho(Q)$.*

Proof. Using the domain of definition $D(L)$ of the operator L and the inequality (3.13), we see that $-\Im_x(\xi u_t)$ and u_t are in $L^2_\rho(Q)$. To prove that $-u_{ttt} \in L^2_\rho(Q)$, we use the t-averaging operators ρ_ε introduced in [5]. Applying the operators ρ_ε and $\partial/\partial t$ to equation (4.2), we obtain

$$\begin{aligned}\|\frac{\partial}{\partial t}\rho_\varepsilon u_{tt}\|^2_{L^2_\rho(Q)} &\le 3\|\frac{\partial}{\partial t}(u-\Im_x(\xi u_t))\|^2_{L^2_\rho(Q)} + 3\|\frac{\partial}{\partial t}\rho_\varepsilon h\|^2_{L^2_\rho(Q)} \\ &\quad + 3\|\frac{\partial}{\partial t}\left[(u-\Im_x(\xi u_t)) - \rho_\varepsilon(u-\Im_x(\xi u_t))\right]\|^2_{L^2_\rho(Q)}.\end{aligned}$$

From this last inequality, it follows that

$$\begin{aligned}\|\frac{\partial}{\partial t}\rho_\varepsilon u_{tt}\|^2_{L^2_\rho(Q)} &\le 6\|u_t\|^2_{L^2_\rho(Q)} + 3\|\frac{\partial}{\partial t}\rho_\varepsilon h\|^2_{L^2_\rho(Q)} + 6\|\Im_x(\xi u_{tt})\|^2_{L^2_\rho(Q)} \\ &\quad + 3\|\frac{\partial}{\partial t}\left[(u-\Im_x(\xi u_t)) - \rho_\varepsilon(u-\Im_x(\xi u_t))\right]\|^2_{L^2_\rho(Q)}.\end{aligned} \tag{4.6}$$

Using the properties of the operators ρ_ε introduced in [5], we deduce from (4.6) that

$$\|\frac{\partial}{\partial t}\rho_\varepsilon u_{tt}\|^2_{L^2_\rho(Q)} \le 6\|u_t\|^2_{L^2_\rho(Q)} + 3\|\frac{\partial}{\partial t}\rho_\varepsilon h\|^2_{L^2_\rho(Q)} + 6\|\Im_x(\xi u_{tt})\|^2_{L^2_\rho(Q)}.$$

Since $\rho_\varepsilon v \underset{\varepsilon\to 0}{\to} v$ in $L^2(Q)$, and $\|\frac{\partial}{\partial t}\rho_\varepsilon u_{tt}\|^2_{L^2_\rho(Q)}$ is bounded, we conclude that Ψ is in $L^2_\rho(Q)$. □

Proof of Theorem 4.1. First, we replace Ψ in (4.1) by its representation (4.5); thus we have

$$\begin{aligned}&\|u_t\|^2_{L^2_\rho(Q)} + (u_t, u_{tt})_{L^2_\rho(Q)} - ((xu_x)_x, u_t)_{L^2(Q)} \\ &- (u_{tt}, u_{ttt})_{L^2_\rho(Q)} - (u_{ttt}, u_t))_{L^2_\rho(Q)} \\ &+ ((xu_x)_x, u_{ttt})_{L^2(Q)} - (u_t, \Im_x(u_{tt}))_{L^2_\rho(Q)} \\ &- (u_{tt}, \Im_x(u_{tt}))_{L^2_\rho(Q)} + ((xu_x)_x, \Im_x(u_{tt}))_{L^2(Q)} = 0.\end{aligned} \tag{4.7}$$

Using conditions (2.4), (2.5) the particular form of u given by the relations (4.2) and (4.4) and integrating by parts each term of (4.7), we obtain

$$(u_t, u_{tt})_{L^2_\rho(Q)} = \frac{1}{2}\|u_t(.,T)\|^2_{L^2_\rho((0,a))}, \tag{4.8}$$

$$-((xu_x)_x, u_t)_{L^2(Q)} = \frac{1}{2}\|u_x(.,T)\|^2_{L^2_\rho((0,a))}, \tag{4.9}$$

$$-(u_{tt}, u_{ttt})_{L^2_\rho(Q)} = \frac{1}{2}\|u_{tt}(.,s)\|^2_{L^2_\rho((0,a))}, \tag{4.10}$$

$$-(u_{ttt}, u_t))_{L^2_\rho(Q)} = \|u_{tt}\|^2_{L^2_\rho(Q_s)}, \tag{4.11}$$

$$((xu_x)_x, u_{ttt})_{L^2(Q)} = \frac{1}{2}\|u_{xt}(.,T)\|^2_{L^2_\rho((0,a))}, \tag{4.12}$$

$$-(u_{tt}, \Im_x(u_{tt}))_{L^2_\rho(Q)} = 0, \tag{4.13}$$

$$((xu_x)_x, \Im_x(u_{tt}))_{L^2(Q)} = -(xu_{tt}, u_x)_{L^2_\rho(Q_s)}. \tag{4.14}$$

Combining equalities (4.7)-(4.14), we get

$$\begin{aligned} &\frac{1}{2}\|u_x(.,T)\|^2_{L^2_\rho((0,a))}+\frac{1}{2}\|u_t(.,T)\|^2_{L^2_\rho((0,a))}+\frac{1}{2}\|u_{tt}(.,s)\|^2_{L^2_\rho((0,a))}\\ &+\|u_{tt}\|^2_{L^2_\rho(Q_s)}+\|u_t\|^2_{L^2_\rho(Q)}+\frac{1}{2}\|u_{xt}(.,T)\|^2_{L^2_\rho((0,a))}\\ &\le (xu_{tt},u_x)_{L^2_\rho(Q_s)}+(u_t,\Im_x(u_{tt}))_{L^2_\rho(Q)}. \end{aligned} \tag{4.15}$$

We then use Young's inequality and (3.13) to estimate the right-hand side of (4.15):

$$(xu_{tt},u_x)_{L^2_\rho(Q_s)}\le 2\|u_{tt}\|^2_{L^2_\rho(Q_s)}+\frac{a^2}{8}\|u_x\|^2_{L^2_\rho(Q_s)}, \tag{4.16}$$

$$(u_t,\Im_x(u_{tt}))_{L^2_\rho(Q)}\le 2\|u_t\|^2_{L^2_\rho(Q_s)}+\frac{a^3}{16}\|u_{tt}\|^2_{L^2_\rho(Q_s)}. \tag{4.17}$$

Hence, inequalities (4.15)-(4.17) yield

$$\begin{aligned} &\|u_x(.,T)\|^2_{L^2_\rho((0,a))}+\|u_t(.,T)\|^2_{L^2_\rho((0,a))}+\|u_{tt}(.,s)\|^2_{L^2_\rho((0,a))}+\|u_{xt}(.,T)\|^2_{L^2_\rho((0,a))}\\ &\le \frac{a^2}{4}\|u_x\|^2_{L^2_\rho(Q_s)}+2\|u_t\|^2_{L^2_\rho(Q_s)}+(\frac{a^3}{8}+2)\|u_{tt}\|^2_{L^2_\rho(Q_s)}\\ &\le d\frac{a^2}{4}\|u_{xt}\|^2_{L^2_\rho(Q_s)}+2\|u_t\|^2_{L^2_\rho(Q_s)}+(\frac{a^3}{8}+2)\|u_{tt}\|^2_{L^2_\rho(Q_s)}\\ &\le \delta\Big(\|u_{xt}\|^2_{L^2_\rho(Q_s)}+\|u_t\|^2_{L^2_\rho(Q_s)}+\|u_{tt}\|^2_{L^2_\rho(Q_s)}\Big), \end{aligned} \tag{4.18}$$

where $d=4(T-s)^2$ is a Poincare constant and $\delta=\max\Big\{d\frac{a^2}{4},\frac{a^3}{8}+2\Big\}$. If we drop the first term on the left-hand side of (4.18), we obtain

$$\begin{aligned} &\|u_t(.,T)\|^2_{L^2_\rho((0,a))}+\|u_{tt}(.,s)\|^2_{L^2_\rho((0,a))}+\|u_{xt}(.,T)\|^2_{L^2_\rho((0,a))}\\ &\le \delta\Big(\|u_{xt}\|^2_{L^2_\rho(Q_s)}+\|u_t\|^2_{L^2_\rho(Q_s)}+\|u_{tt}\|^2_{L^2_\rho(Q_s)}\Big). \end{aligned} \tag{4.19}$$

Now we define a new unknown function $\theta(x,t)$ by $\theta_t(x,t)=-u_{tt}$, such that $\theta(x,T)=0$; that is,

$$\theta(x,t)=\int_t^T u_{ss}ds.$$

Then we have

$$u_t(x,t)=\theta(x,s)-\theta(x,t)\quad\text{and}\quad u_t(x,T)=\theta(x,s).$$

Thus inequality (4.19) can be written as

$$\begin{aligned} &\|u_{tt}(.,s)\|^2_{L^2_\rho((0,a))}+\|\theta_x(x,s)\|^2_{L^2_\rho((0,a))}+\|\theta(x,s)\|^2_{L^2_\rho((0,a))}\\ &\le \delta\int_s^T\Big\{\int_0^a x(\theta(x,s)-\theta(x,t))^2dx+\int_0^a xu_{tt}^2dx\\ &\quad+\int_0^a x(\theta_x(x,s)-\theta_x(x,t))^2dx\Big\}dt. \end{aligned} \tag{4.20}$$

It follows from (4.20) that

$$\begin{aligned} &(1-2\delta(T-s)\Big(\|\theta_x(x,s)\|^2_{L^2_\rho((0,a))}+\|\theta(x,s)\|^2_{L^2_\rho((0,a))}\Big)+\|u_{tt}(.,s)\|^2_{L^2_\rho((0,a))}\\ &\le 2\delta\Big(\|u_{tt}\|^2_{L^2_\rho(Q_s)}+\|\theta_x\|^2_{L^2_\rho(Q_s)}+\|\theta\|^2_{L^2_\rho(Q_s)}\Big). \end{aligned} \tag{4.21}$$

If $s_0 > 0$ satisfies $T - s_0 = 1/4$, then (4.21) implies

$$\begin{aligned} &\|u_{tt}(.,s)\|^2_{L^2_\rho((0,a))} + \|\theta_x(x,s)\|^2_{L^2_\rho((0,a))} + \|\theta(x,s)\|^2_{L^2_\rho((0,a))} \\ &\le 4\delta \left(\|u_{tt}\|^2_{L^2_\rho(Q_s)} + \|\theta_x\|^2_{L^2_\rho(Q_s)} + \|\theta\|^2_{L^2_\rho(Q_s)} \right), \end{aligned} \tag{4.22}$$

for all $s \in [T - s_0, T]$. Inequality (4.22) in turns implies that

$$-\sigma'(s) \le 4\delta\sigma(s), \tag{4.23}$$

where

$$\sigma(s) = \|u_{tt}\|^2_{L^2_\rho(Q_s)} + \|\theta_x\|^2_{L^2_\rho(Q_s)} + \|\theta\|^2_{L^2_\rho(Q_s)}.$$

Since $\sigma(T) = 0$, then an integration of (4.23) over $[s, T]$ gives

$$\sigma(s)e^{4\delta s} \le 0, \quad \forall s \in [T - s_0, T].$$

It follows from the above inequality that $\Psi \equiv 0$ almost everywhere on the domain $Q_{T-s_0} = (0, a) \times [T - s_0, T]$. The length s does not depend on the origin, so we can proceed in the same way a finite number of times to show that $\Psi \equiv 0$ in Q. □

Theorem 4.3. *The range of $R(L)$ of the operator L coincides with F.*

Proof. Suppose that for some $W = (\Psi, \Psi_1, \Psi_2) \in R(L)^\perp$,

$$(\mathcal{L}u, \Psi)_{L^2_\rho(Q)} + (\ell_1 u, \Psi_1)_{V^{1,0}_\rho((0,a))} + (\ell_2 u, \Psi_2)_{L^2_\rho((0,a))} = 0 \tag{4.24}$$

We must prove that $W = 0$. Let $D_0(L) = \{u/u \in D(L) : \ell_1 u = \ell_2 u = 0\}$, and put $u \in D_0(L)$ in (4.24),we get

$$(\mathcal{L}u, \Psi)_{L^2_\rho(Q)} = 0, \quad \forall u \in D(L).$$

Hence, by theorem 4.1 it follows that $\Psi = 0$. Thus (4.24) becomes

$$(\ell_1 u, \Psi_1)_{V^{1,0}_\rho((0,a))} + (\ell_2 u, \Psi_2)_{L^2_\rho((0,a))} = 0. \tag{4.25}$$

Since $\ell_1 u$ and $\ell_2 u$ are independent and the ranges of the operators ℓ_1 and ℓ_2 are everywhere dense in the spaces $V^{1,0}_\rho((0, a))$ and $L^2_\rho((0, a))$ respectively. Hence the inequality (4.25) implies that $\Psi_1 = \Psi_2 = 0$. Consequently $W = 0$. This completes the proof. □

5. The semilinear problem

In this section we state and prove the existence of a local solution to problem (1.1). First, we state some lemmas.

Lemma 5.1. *For any v in $V^{1,0}_\rho((0, a))$ satisfying the boundary condition (2.4), we have*

$$\int_0^a xv^2(x)dx \le 4a^2 \int_0^a x(v_x(x))^2 dx. \tag{5.1}$$

Proof. It is easy to see that for each smooth function v satisfying the boundary condition (2.4), we have

$$0 = \int_0^a (xv^2)_x dx = \int_0^a (v^2 + 2xvv_x)dx;$$

hence,

$$\int_0^a xv^2 dx \le a \int_0^a v^2 dx = -2 \int_0^a xvv_x dx.$$

Using Young's inequality we obtain

$$\int_0^a xv^2dx \le |2\int_0^a xvv_x dx| \le 2a^2\int_0^a xv_x^2dx + \frac{1}{2}\int_0^a xv^2dx.$$

Therefore (5.1) is established for any smooth function v. This inequality remains valid for v in $V_\rho^{1,0}((0,a))$ by a density argument. □

Lemma 5.2. *For v in $V_\rho^{1,0}((0,a))$ satisfying the boundary condition (2.4) and $2 < p < 3$, we have $|v|^{p-2}v \in L_\rho^2((0,a))$.*

Proof. First we note that by virtue of lemma 5.42 of [1] and by using a density argument we have

$$\sup_{0\le x\le a} x(v(x))^2 \le 4\int_0^a xv^2(x)dx + 4\int_0^a x|v(x)\|v'(x)|dx. \tag{5.2}$$

Using the Schwarz inequality and lemma 5.1, estimate (5.2) yields

$$\sup_{0\le x\le a} x(v(x))^2 \le C\int_0^a x|v'(x)|^2dx. \tag{5.3}$$

Evaluating the L_ρ^2-norm of $|v|^{p-2}v$ we have

$$\begin{aligned}\int_0^a x|v(x)|^{2p-2}dx &= \int_0^a x^{p-1}|v(x)|^{2(p-1)}x^{2-p}dx\\ &\le \Big(\sup_{0\le x\le a} x(v(x))^2\Big)^{p-1}\int_0^a x^{2-p}dx\\ &\le \frac{C}{3-p}\Big(\|v\|_{V_\rho^{1,0}((0,a))}\Big)^{2p-2} < \infty,\end{aligned} \tag{5.4}$$

by virtue of (5.3). This completes the proof. □

Theorem 5.3. *If $2 < p < 3$ then for any ϕ in $V_\rho^{1,0}((0,a))$ and ψ in $L_\rho^2((0,a))$, problem (1.1) has a unique local solution $u \in E$.*

Proof. We prove this theorem by using a fixed point argument. For $T > 0$ and $M > 0$, we define the class of functions $W = W(M,T)$, which consists of all functions $w \in E$ satisfying conditions (2.3)-(2.5) and for which we have $\|w\|_E \le M$. We then define a map $h : W \to E$ which associates to each $v \in W$ the solution u of the linear problem

$$\begin{gathered}u_{tt} + u_t - \frac{1}{x}(xu_x)_x = |v|^{p-2}v\\ u(a,t) = 0, \quad \int_0^a xu(x,t)dx = 0\\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x).\end{gathered} \tag{5.5}$$

It follows from theorem 3.1 and theorem 4.3 that (5.5) has a unique solution u satisfying

$$\|u\|_E^2 \le C\Big\{\|\phi\|^2_{V_\rho^{1,0}((0,a))} + \|\psi\|^2_{L_\rho^2((0,a))} + \||v|^{p-2}v\|^2_{L_\rho^2((Q)}\Big\}.$$

This, in turn, implies by (5.4) that

$$\begin{aligned} \|u\|_E^2 &\le C\Big\{\|\phi\|^2_{V_\rho^{1,0}((0,a))} + \|\psi\|^2_{L^2_\rho((0,a))} + \int_0^T (\|v\|_{V_\rho^{1,0}((0,a))})^{2p-2} dt\Big\} \\ &\le C\left\{\|\phi\|^2_{V_\rho^{1,0}((0,a))} + \|\psi\|^2_{L^2_\rho((0,a))} + CT\|v\|_E^{2p-2}\right\} \\ &\le C\left\{\|\phi\|^2_{V_\rho^{1,0}((0,a))} + \|\psi\|^2_{L^2_\rho((0,a))} + CTM^{2p-2}\right\} \end{aligned} \tag{5.6}$$

Taking M so large that $C\{\|\phi\|^2_{V_\rho^{1,0}((0,a))} + \|\psi\|^2_{L^2_\rho((0,a))}\} \le M^2/2$ and T so small that $CTM^{2p-2} \le M^2/2$, estimate (5.6) yields

$$\|u\|_E^2 \le M^2;$$

hence h maps W into itself. To show that h is a contraction for T small enough, we consider v_1, $v_2 \in W$ and the corresponding images u_1 and u_2. It is straightforward to see that $U = u_1 - u_2$ satisfies

$$\begin{gathered} U_{tt} + U_t - \frac{1}{x}(xU_x)_x = |v_1|^{p-2}v_1 - |v_2|^{p-2}v_2 \\ U(a,t) = 0, \quad \int_0^a xU(x,t)dx = 0 \\ U(x,0) = 0, \quad U_t(x,0) = 0. \end{gathered} \tag{5.7}$$

We multiply (5.7) by xU_t and integrate over Q to get

$$\begin{aligned} &\frac{1}{2}\int_0^a x{U_t}^2(x,t)dx + \frac{1}{2}\int_0^a x{U_x}^2(x,t)dx + \int_0^t\int_0^a xU_t^2(x,s)\,dx\,ds \\ &\le \int_0^t\int_0^a x|U_t|\,||v_1|^{p-2}v_1 - |v_2|^{p-2}v_2|(x,s)\,dx\,ds\,. \end{aligned}$$

Schwarz inequality then leads to

$$\begin{aligned} &\int_0^a x{U_t}^2(x,t)dx + \int_0^a x{U_x}^2(x,t)dx + \int_0^t\int_0^a xU_t^2(x,s)\,dx\,ds \\ &\le \int_0^t\int_0^a x\{|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\}^2(x,s)\,dx\,ds. \end{aligned} \tag{5.8}$$

We now estimate the right-hand-side of (5.8) as follows. Taking $V = v_1 - v_2$, we obtain

$$\int_0^a x\{|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\}^2 dx \le C_1\int_0^a x|V|^2\{|v_1|^{2p-4} + |v_2|^{2p-4}\}, \tag{5.9}$$

where C_1 is a constant independent of v_1, v_2 and t. Thus we have, by virtue of (5.3),

$$\begin{aligned} \int_0^a x\{|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\}^2 dx &\le C_1 \sup_{0\le x\le a} x(V(x))^2 \int_0^a \{|v_1|^{2p-4} + |v_2|^{2p-4}\}dx \\ &\le C\big(\int_0^a x|V_x|^2 dx\big)\int_0^a \{|v_1|^{2p-4} + |v_2|^{2p-4}\}dx. \end{aligned}$$

Next we evaluate

$$\begin{aligned}\int_0^a |v_1|^{2p-4}dx &= \int_0^a x^{p-2}|v_1|^{2p-4}x^{2-p}dx\\ &\le \big(\sup_{0\le x\le a} x|v_1|^2\big)^{p-2}\int_0^a x^{2-p}dx\\ &\le \frac{C}{3-p}\Big[\int_0^1 x(\frac{\partial v_1}{\partial x})^2dx\Big]^{p-2} \le CM^{2(p-2)}.\end{aligned}$$

By combining (5.8),(5.9), we arrive at

$$\int_0^{T^*}\int_0^a x\{|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\}^2dxds \le CTM^{2(p-2)}\|V\|_E^2. \tag{5.10}$$

Therefore (5.8) and (5.10) give

$$\|U\|_E^2 \le CTM^{2(p-2)}\|V\|_E^2. \tag{5.11}$$

Choosing T small enough that $CTM^{2(p-2)} < 1$, makes the map h a contraction from W into itself. The Contraction Mapping Theorem then guarantees the existence of a fixed point u, which is the desired solution of (1.1). The proof is then completed. □

6. Finite time blow up

In this section we show that the solution of (1.1) blows up in finite time if

$$\mathcal{E}_0 := \frac{1}{2}\int_0^a x(\psi(x))^2dx + \frac{1}{2}\int_0^a x(\phi_x(x))^2dx - \frac{1}{p}\int_0^a x|\phi(x)|^pdx < 0. \tag{6.1}$$

Theorem 6.1. *If $2 < p < 3$ then for any ϕ in $V_\rho^{1,0}((0,a))$ and ψ in $L_\rho^2((0,a))$ satisfying (2.4), (2.5), and (6.1), the solution of problem (1.1) blows up in finite time.*

Proof. We define the functional

$$\mathcal{E}(t) := \frac{1}{2}\int_0^a x(u_t(x,t))^2dx + \frac{1}{2}\int_0^a x(u_x(x,t))^2dx - \frac{1}{p}\int_0^a x|u(x,t)|^pdx. \tag{6.2}$$

Multiplying (1.1) by xu_t and integrating over $(0,a)$ yields

$$\mathcal{E}^{'}(t) = -\int_0^a xu_t^2(x,t)dx \le 0; \tag{6.3}$$

hence $\mathcal{E}(t) \le \mathcal{E}_0(0) < 0$, for all $t \ge 0$. By setting $H(t) = -\mathcal{E}(t)$, we get

$$0 < H(0) \le H(t) \le \frac{1}{p}\int_0^a x\,|u(x,t)|^p\,dx, \quad \forall t \ge 0. \tag{6.4}$$

Then we define

$$L(t) := H^{2/p}(t) + \varepsilon\int_0^a xuu_t(x,t)dx + \frac{\varepsilon}{2}\int_0^a xu^2(x,t)dx \tag{6.5}$$

for ε small enough so that

$$L(0) = H^{2/p}(0) + \varepsilon\int_0^a x\phi\psi(x)dx + \frac{\varepsilon}{2}\int_0^a x\phi^2(x)dx > 0$$

By differentiating (6.5) and using (1.1) and (6.2), we obtain

$$\begin{aligned}
L'(t) &= \frac{2}{p} H^{-1+2/p}(t)\, H'(t) + \varepsilon \int_0^a x u_t^2(x,t)dx \\
&\quad + \varepsilon \int_0^a x u u_{tt}(x,t)dx + \varepsilon \int_0^a x u u_t(x,t)dx \\
&= \frac{2}{p} H^{-1+2/p}(t)\, H'(t) + \varepsilon \int_0^a x u_t^2(x,t)dx + \varepsilon \int_0^a x u u_t(x,t)dx \\
&\quad + \varepsilon \int_0^a x u[-u_t + \frac{1}{x}(x u_x)_x + |u|^{p-2}u]dx \\
&\geq \varepsilon \int_0^a x u_t^2(x,t)dx - \varepsilon \int_0^a x(u_x(x,t))^2 dx + \varepsilon \int_0^a x|u(x,t)|^p dx \\
&= \varepsilon \int_0^a x u_t^2(x,t)dx - \varepsilon \int_0^a x(u_x(x,t))^2 dx \\
&\quad + \varepsilon(1-\frac{2}{p}) \int_0^a x|u(x,t)|^p dx \\
&\quad + \frac{2\varepsilon}{p}[pH(t) + \frac{p}{2}\int_0^a x u_t^2(x,t)dx + \frac{p}{2}\int_0^a x(u_x(x,t))^2 dx] \\
&= 2\varepsilon \int_0^a x u_t^2(x,t)dx + 2\varepsilon H(t) + \varepsilon(1-\frac{2}{p}) \int_0^a x|u(x,t)|^p dx \\
&= \varepsilon(1-\frac{2}{p})\Big(H(t) + \int_0^a x|u(x,t)|^p dx + \int_0^a x u_t^2(x,t)dx\Big).
\end{aligned} \tag{6.6}$$

The next estimate reads

$$\Big[\int_0^a x u^2 dx\Big]^{p/2} \leq \Big[(\int_0^a x|u|^p dx)^{2/p}(\int_0^a x dx)^{(p-2)/p}\Big]^{p/2} \leq (\frac{a^2}{2})^{(p-2)/2}\int_0^a x|u|^p dx \tag{6.7}$$

and

$$\begin{aligned}
|\int_0^a x u u_t dx| &\leq \Big(\int_0^a x u^2 dx\Big)^{1/2}\Big(\int_0^a x u_t^2 dx\Big)^{1/2} \\
&\leq (\frac{a^2}{2})^{(p-2)/2p}\Big(\int_0^a x|u|^p dx\Big)^{1/p}\Big(\int_0^a x u_t^2 dx\Big)^{1/2},
\end{aligned}$$

which implies

$$\Big|\int_0^a x u u_t dx\Big|^{p/2} \leq (\frac{a^2}{2})^{(p-2)/4}\Big(\int_0^a x|u|^p dx\Big)^{1/2}\Big(\int_0^a x u_t^2 dx\Big)^{p/4}.$$

Also Young's inequality gives

$$\Big|\int_0^a x u u_t dx\Big|^{p/2} \leq C\Big[\Big(\int_0^a x|u|^p dx\Big)^{\mu/2} + \Big(\int_0^a x u_t^2 dx\Big)^{\theta p/4}\Big]$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 8/p$, (hence $\mu = 8/(8-p)$) to get

$$\Big|\int_0^a x u u_t dx\Big|^{p/2} \leq C\Big[\Big(\int_0^a x|u|^p dx\Big)^{4/(8-p)} + \int_0^a x u_t^2 dx\Big].$$

Using that $z^\nu \le z+1 \le (1+\frac{1}{a})(z+a)$ for all $z \ge 0$, $0<\nu\le 1$, $a \ge 0$, we have the following estimate

$$\begin{aligned}\Big(\int_0^a x|u|^p dx\Big)^{4/(8-p)} &\le \big(1+\frac{1}{H(t)}\big)\Big(\int_0^a x|u|^p dx + H(t)\Big)\\ &\le \big(1+\frac{1}{H(0)}\big)\Big(\int_0^a x|u|^p dx + H(t)\Big)\end{aligned} \tag{6.8}$$

Consequently,

$$\Big|\int_0^a xuu_t dx\Big|^{p/2} \le C\Big[\int_0^a x|u|^p dx + H(t) + \int_0^a xu_t^2 dx\Big]. \tag{6.9}$$

A combination of (6.5), (6.7), and (6.9) leads to

$$L^{p/2}(t) \le C\Big[\int_0^a x|u|^p dx + H(t) + \int_0^a xu_t^2 dx\Big]. \tag{6.10}$$

Therefore, using (6.6) and (6.10), we obtain

$$L'(t) \ge \lambda L^{p/2}(t) \tag{6.11}$$

where λ is a constant depending only on ε, $H(0)$, and a. Integration of (6.11) over $(0,t)$ gives

$$L^{(p/2)-1}(t) \ge \frac{1}{L^{1-(p/2)}(0) - \lambda(p/2-1)t};$$

hence $L(t)$ blows up in a time

$$T^* \le \frac{1}{\lambda(p/2-1)L^{1-(p/2)}(0)}. \tag{6.12}$$

□

Remark The time estimate (6.12) shows that the larger $L(0)$ is the quicker the blow up takes place.

7. Decay of Solutions

In this section we show that any solution of (1.1) is global and decays exponentially provided that $\mathcal{E}_0$ is positive and small enough. In order to state and prove our results we introduce the following:

$$\begin{gathered} I(t) = I(u(t)) = \int_0^a xu_x^2 dx - \int_0^a x|u|^p dx \\ J(t) = J(u(t)) = \frac{1}{2}\int_0^a xu_x^2 dx - \frac{1}{p}\int_0^a x|u|^p dx \\ \mathcal{H} = \{w \in V_\rho^{1,0}((0,a)) : I(w) > 0\} \cup \{0\} \end{gathered}$$

Remark Note that $\mathcal{E}(t) = J(t) + \frac{1}{2}\int_0^a xu_t^2 dx$.

Lemma 7.1. *For v in $V_\rho^{1,0}((0,a))$ satisfying the boundary condition (2.4) and for $2 \le p < 4$, we have*

$$\int_0^a x|v|^p dx \le C_* \|v_x\|^p_{L^2_\rho((0,a))}, \tag{7.1}$$

where C_* is a constant depending on a and p only.

Proof. A direct calculation, using (5.3), gives

$$\begin{aligned} \int_0^a x|v|^p dx &= \int_0^a \left(x|v|^2\right)^{p/2} x^{1-p/2} dx \\ &\le \Big(\sup_{0\le x\le a} x|v|^2\Big)^{p/2} \int_0^a x^{1-p/2} dx \\ &\le \Big(C\int_0^a x|v'(x)|^2 dx\Big)^{p/2} \int_0^a x^{1-p/2} dx \\ &= C_* \|v_x\|^p_{L^2_\rho((0,a))}. \end{aligned} \tag{7.2}$$

□

Lemma 7.2. *Suppose that $2 < p < 3$ and $\phi \in H$, $\psi \in L^2_\rho((0,a))$ satisfying (2.4), (2.5), and*

$$\beta = C_*\big(\frac{2p}{p-2}\mathcal{E}_0\big)^{(p-2)/2} < 1. \tag{7.3}$$

Then $u(t) \in \mathcal{H}$ for each $t \in [0,T)$.

Proof. Since $I(u_0) > 0$ then there exists $T_m \le T$ such that $I(u(t)) \ge 0$ for all $t \in [0,T_m)$. This implies

$$\begin{aligned} J(t) &= \frac12\int_0^a xu_x^2 dx - \frac1p\int_0^a x|u|^p dx \\ &= \frac{p-2}{2p}\int_0^a xu_x^2 dx + \frac1p I(u(t)) \\ &\ge \frac{p-2}{2p}\int_0^a xu_x^2 dx, \qquad \forall t\in[0,T_m); \end{aligned} \tag{7.4}$$

hence

$$\int_0^a xu_x^2 dx \le \frac{2p}{p-2}J(t) \le \frac{2p}{p-2}\mathcal{E}(t) \le \frac{2p}{p-2}\mathcal{E}_0, \qquad \forall t\in[0,T_m). \tag{7.5}$$

Using (7.1), (7.3), and (7.5), we easily arrive at

$$\begin{aligned} \int_0^a x|u|^p dx &\le C_*\|u_x\|^p_{L^2_\rho((0,a))} = C_*\|u_x\|^{p-2}_{L^2_\rho((0,a))}\|u_x\|^2_{L^2_\rho((0,a))} \\ &\le C_*\left(\frac{2p}{p-2}\mathcal{E}_0\right)^{(p-2)/2}\|u_x\|^2_{L^2_\rho((0,a))} = \beta\|u_x\|^2_{L^2_\rho((0,a))} \\ &< \|u_x\|^2_{L^2_\rho((0,a))}, \quad \forall t\in[0,T_m); \end{aligned} \tag{7.6}$$

hence

$$\|u_x\|^2_{L^2_\rho((0,a))} - \int_0^a x|u|^p dx > 0, \forall t\in[0,T_m).$$

This shows that $u(t) \in \mathcal{H}, \forall t \in [0,T_m)$. By repeating the procedure, T_m is extended to T. □

Theorem 7.3. *Suppose that $2 < p < 3$ and $\phi \in \mathcal{H}$, $\psi \in L^2_\rho((0,a))$ satisfying (2.4), (2.5), and (7.3). Then the solution of problem (1.1) is a global solution.*

Proof. It suffices to show that $\|u_x\|^2_{L^2_\rho((0,a))}+\|u_t\|^2_{L^2_\rho((0,a))}$ is bounded independently of t. To achieve this we use (6.3); so we have

$$\begin{aligned}\mathcal{E}_0 \geq \mathcal{E}(t) &= \frac{1}{2}\|u_x\|^2_{L^2_\rho((0,a))} - \frac{1}{p}\int_0^a x|u|^p dx + \frac{1}{2}\|u_t\|^2_{L^2_\rho((0,a))} \\ &= \frac{p-2}{2p}\|u_x\|^2_{L^2_\rho((0,a))} + \frac{1}{p}I(u(t)) + \frac{1}{2}\|u_t\|^2_{L^2_\rho((0,a))} \\ &\geq \frac{p-2}{2p}\|u_x\|^2_{L^2_\rho((0,a))} + \frac{1}{2}\|u_t\|^2_{L^2_\rho((0,a))}\end{aligned} \tag{7.7}$$

since $I(u(t)) \geq 0$. Therefore,

$$\|u_x\|^2_{L^2_\rho((0,a))} + \|u_t\|^2_{L^2_\rho((0,a))} \leq \frac{2p}{p-2}\mathcal{E}_0.$$

□

Theorem 7.4. *Suppose that $2 < p < 3$ and $\phi \in \mathcal{H}$, $\psi \in L^2_\rho((0,a))$ satisfying (2.4), (2.5), and (7.3). Then there exist positive constants K and k such that the global solution of problem (1.1) satisfies*

$$\mathcal{E}(t) \leq Ke^{-kt}, \quad \forall t \geq 0. \tag{7.8}$$

Proof. We define

$$\mathcal{F}(t) := \mathcal{E}(t) + \varepsilon\int_0^a x\big(uu_t + \frac{1}{2}u^2\big)dx, \tag{7.9}$$

for ε small such that

$$a_1\mathcal{F}(t) \leq \mathcal{E}(t) \leq a_2\mathcal{F}(t), \tag{7.10}$$

holds for two positive constants a_1 and a_2. This is, of course possible by (5.1) and (7.5). We differentiate (7.9) and use equation (1.1) to obtain

$$\begin{aligned}\mathcal{F}'(t) &= -\int_0^a x|u_t|^2 dx + \varepsilon\int_0^a x[u_t^2 - |u_x|^2 + |u(t)|^p]dx \\ &\leq -[1-\varepsilon]\int_0^a x|u_t|^2 dx - \varepsilon\int_0^a x|u_x|^2 dx + \varepsilon\int_0^a x|u(t)|^p dx.\end{aligned} \tag{7.11}$$

We then use (6.2) and (7.6) to get

$$\begin{aligned}\int_0^a x|u|^p dx &= \alpha\int_0^a x|u|^p dx + (1-\alpha)\int_0^a x|u|^p dx \\ &\leq \alpha\Big(\frac{p}{2}\int_0^a xu_t^2 dx + \frac{p}{2}\int_0^a xu_x^2 dx - p\mathcal{E}(t)\Big) \\ &\quad + (1-\alpha)\beta\int_0^a xu_x^2 dx, \quad 0 < \alpha < 1\end{aligned} \tag{7.12}$$

Therefore, a combination of (7.11) and (7.12) gives

$$\mathcal{F}'(t) \leq -[1-\varepsilon(\frac{\alpha p}{2}+1)]\int_\Omega u_t^2(t)dx - \alpha p\mathcal{E}(t) + \varepsilon[\alpha(\frac{p}{2}-1) - \eta(1-\alpha)]\int_0^a xu_x^2 dx \tag{7.13}$$

where $\eta = 1-\beta$. By using (7.5) and choosing α close to 1 so that $\alpha(\frac{p}{2}-1)-\eta(1-\alpha) \geq 0$, estimate (7.13) takes the form

$$\begin{aligned} \mathcal{F}'(t) &\leq -[1-\varepsilon(\frac{\alpha p}{2}+1)]\int_\Omega u_t^2(t)dx - \alpha p\mathcal{E}(t) \\ &\quad + \varepsilon[\alpha(\frac{p}{2}-1)-\eta(1-\alpha)]\frac{2p}{p-2}\mathcal{E}(t) \\ &\leq -[1-\varepsilon(\frac{\alpha p}{2}+1)]\int_\Omega u_t^2(t)dx - \eta\varepsilon(1-\alpha)\frac{2p}{p-2}\mathcal{E}(t). \end{aligned} \tag{7.14}$$

At this point we choose ε so small that $1-\varepsilon(\frac{\alpha p}{2}+1) \geq 0$, and (7.10) remains valid. Consequently (7.14) yields

$$\mathcal{F}'(t) \leq -\eta\varepsilon(1-\alpha)\frac{2p}{p-2}\mathcal{E}(t) \leq -\varepsilon a_2\eta(1-\alpha)\frac{2p}{p-2}\mathcal{F}(t) \tag{7.15}$$

by virtue of (7.10). A simple integration of (7.15) leads to

$$\mathcal{F}(t) \leq \mathcal{F}(0)e^{-kt},$$

where $k = \varepsilon a_2[\eta(1-\alpha)\frac{2p}{p-2}]$. Again using (7.10), we obtain (7.8). This completes the proof. □

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