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Note

A note on blow up of solutions of a quasilinear heat equation with vanishing initial energy

Salim A. Messaoudi

Mathematical Sciences Department, KFUPM, Dhahran 31261, Saudi Arabia

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Abstract

In this work we consider an initial boundary value problem related to the equation

$$u_t - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) = f(u)$$

and prove, under suitable conditions on f, a blow up result for solutions with vanishing or negative initial energy.

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1. Introduction

In this paper we are concerned with the finite time blow up of solutions for the initial boundary value problem

$$\begin{aligned} &u_t - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) = f(u), \quad x \in \Omega,\ t > 0,\\ &u(x,t) = 0, \quad x \in \partial\Omega,\ t \geqslant 0,\\ &u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{1.1}$$

where $m > 2$, f is a continuous function, and Ω is a bounded domain of $\mathbb{R}^n$ ($n \geqslant 1$), with a smooth boundary $\partial\Omega$.

In 1993, Junning [2] studied (1.1) and established a global existence result for f depending on u as well as on ∇u. He also proved a nonglobal existence result for (1.1) under the condition

E-mail address: messaoud@kfupm.edu.sa.

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$$\frac{1}{m}\int_{\Omega}\left|\nabla u_0(x)\right|^m dx - \int_{\Omega} F\left(u_0(x)\right) dx \leqslant -\frac{4(m-1)}{mT(m-2)^2}\int_{\Omega} u_0^2(x)\,dx, \tag{1.2}$$

where $F(u)=\int_0^u f(s)\,ds$. More precisely, he showed that if there exists $T>0$, for which (1.2) holds, then the solution blows up in a time less than T. This type of results have been extensively generalized and improved by Levine et al. in [3], where the authors proved some global, as well as nonglobal, existence theorems. Their result, when applied to problem (1.1), requires that

$$\frac{1}{m}\int_{\Omega}\left|\nabla u_0(x)\right|^m dx - \int_{\Omega} F\left(u_0(x)\right) dx < 0. \tag{1.3}$$

We note that the inequality (1.3) implies (1.2). In 1999, Erdem [1] discussed the initial Dirichlet-type boundary problem for

$$\begin{aligned} &u_t - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left(d+|\nabla u|^{m-2}\right)\frac{\partial u}{\partial x_i}\right) + g(u,\nabla u) = f(u),\\ &\quad x\in\Omega,\ t>0. \end{aligned} \tag{1.4}$$

He established a blow up result, under a condition similar to (1.3) and another one on the growth of g.

Concerning global existence, Nakao and Chen [5] studied the following problem:

$$\begin{aligned} &u_t - \operatorname{div}\left(\sigma\left(|\nabla u|^2\right)\nabla u\right) + b(u)\nabla u = 0, \quad x\in\Omega,\ t>0,\\ &u(x,t)=0, \quad x\in\partial\Omega,\ t\geqslant 0,\\ &u(x,0)=u_0(x), \quad x\in\Omega, \end{aligned} \tag{1.5}$$

where $\sigma(v)$ behaves like $|v|^m$, $m\geqslant 0$, and $|b(u)|\leqslant k_0|u|^{\beta}$, $k_0>0$, $\beta\geqslant 0$. He proved global existence, derived precise estimates for $\nabla u(t)$, and showed that solutions decay as $t\to\infty$. His work improves an earlier one by Nakao and Ohara [4], in which he considered (1.5) with $b\equiv 0$.

It is also worth mentioning that Nakao and Ohara [6] considered the periodic solutions of (1.5), with the last term replaced by $g(x,u)-f(x,t)$. He showed that these periodic solutions belong to $L^{\infty}(\omega, W^{1,\infty}(\Omega))$ and gave a bound of $\|\nabla u(t)\|_{\infty}$ under certain geometric conditions on $\partial\Omega$.

Here we show that the blow up can be obtained even for vanishing energy. More precisely, we will get a blow up under the condition

$$\frac{1}{m}\int_{\Omega}\left|\nabla u_0(x)\right|^m dx - \int_{\Omega} F\left(u_0(x)\right) dx \leqslant 0. \tag{1.6}$$

To make this paper self-contained we state, without proof, the local existence result of [2].

Proposition. *Let f be in $C(\mathbb{R})$ satisfying*

$$\big|f(u)\big| \leqslant g(u) \tag{1.7}$$

for g a C^1 function. Then for any $u_0 \in L^\infty(\Omega) \cap H_0^m(\Omega)$, the problem (1.1) *has a solution*

$$\begin{aligned} &u \in L^\infty\big(\Omega \times (0,T)\big) \cap L^m\big((0,T); H_0^m(\Omega)\big), \\ &u_t \in L^2\big(\Omega \times (0,T)\big). \end{aligned} \tag{1.8}$$

2. Blow up

In this section we state and prove our main result.

Theorem. *Let f be in $C(\mathbb{R})$ satisfying* (1.7) *and*

$$pF(u) \leqslant uf(u), \quad p > m > 2. \tag{2.1}$$

Then for any nonzero $u_0 \in L^\infty(\Omega) \cap H_0^m(\Omega)$ satisfying (1.6)*, the solution* (1.8) *blows up in finite time.*

Remark. An example of a function f satisfying (2.1) is $f(s) = |s|^{p-2}s$, for $p > m > 2$. This shows that, in a sense, the source has to dominate the m-Laplacian term.

Proof. We define

$$H(t) = \int_\Omega F\big(u(x,t)\big)\,dx - \frac{1}{m}\int_\Omega \big|\nabla u(x,t)\big|^m\,dx.$$

By using (1.1), we easily arrive at

$$H'(t) = \int_\Omega u_t^2(x,t)\,dx \geqslant 0;$$

hence $H(t) \geqslant H(0) \geqslant 0$, by virtue of (1.6). We then set

$$L(t) = \frac{1}{2}\int_\Omega u^2(x,t)\,dx$$

and differentiate L to get

$$L'(t) = \int_\Omega uu_t(x,t)\,dx \geqslant \int_\Omega u\big[\operatorname{div}\big(|\nabla u|^{m-2}\nabla u\big) + f(u)\big](x,t)\,dx$$

$$\geqslant pH(t) + \left(\frac{p}{m} - 1\right) \int_{\Omega} |\nabla u|^m (x,t)\, dx$$

$$\geqslant \left(\frac{p}{m} - 1\right) \left[H(t) + \|\nabla u\|_m^m\right] \geqslant 0. \tag{2.2}$$

Next we estimate $L^{m/2}(t)$:

$$L^{m/2}(t) \leqslant C\|u\|_m^m \leqslant C\|\nabla u\|_m^m,$$

by Poincare's inequality and the embedding of the L^q spaces. Here C is a constant depending on Ω and m only. Therefore we have

$$L^{m/2}(t) \leqslant C\left[H(t) + \|\nabla u\|_m^m\right]. \tag{2.3}$$

By combining (2.2) and (2.3) we have

$$L'(t) \geqslant \gamma L^{m/2}(t), \tag{2.4}$$

where $\gamma = (p-m)/Cm$. A direct integration of (2.4) then yields

$$L^{m/2-1}(t) \geqslant \frac{1}{L^{1-m/2}(0) - \gamma t}.$$

Therefore L blows up in a time $t^* \leqslant 1/\gamma L^{(m/2)-1}(0)$. □

Corollary. *If there exists $t_0 \geqslant 0$, for which*

$$\frac{1}{m} \int_{\Omega} \left|\nabla u(x,t_0)\right|^m dx - \int_{\Omega} F\left(u(x,t_0)\right) dx = 0,$$

then the solution (1.8) *either remains equal to zero for all time $t \geqslant t_0$ or blows up in finite time $t^* > t_0$.*

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References

[1] D. Erdem, Blow-up of solutions to quasilinear parabolic equations, Appl. Math. Lett. 12 (1999) 65–69.
[2] Z. Junning, Existence and nonexistence of solutions for $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(\nabla u, u, x, t)$, J. Math. Anal. Appl. 172 (1993) 130–146.
[3] H. Levine, S. Park, J. Serrin, Global existence and nonexistence theorems for quasilinear evolution equations of formally parabolic type, J. Differential Equations 142 (1998) 212–229.
[4] M. Nakao, Y. Ohara, Gradient estimates for a quasilinear parabolic equation of the mean curvature type, J. Math. Soc. Japan 48 (1996) 455–466.

[5] M. Nakao, C. Chen, Global existence and gradient estimates for the quasilinear parabolic equations of m-Laplacian type with a nonlinear convection term, J. Differential Equations 162 (2000) 224–250.
[6] M. Nakao, Y. Ohara, Gradient estimates of periodic solutions for quasilinear parabolic equations, J. Math. Anal. Appl. 204 (1996) 868–883.