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LOW UP IN SOLUTIONS OF
A SEMILINEAR WAVE EQUATION Received: May 30, 1999  BLOW UP IN SOLUTIONS OF ... Salim A. Messaoudi
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Abstract We consider the following semilinear wave equation

$$u_{tt}(x,t) - \Delta u(x,t) = -m u_t(x,t) + \bigtriangledown\phi(x) \cdot \bigtriangledown u(x,t) + f(u(x,t))$$

and show that weak solutions blow up in finite time even for small initial data.

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**Key Words**: wave equation, solutions, blow up, soliton energy, decay

## 1. Introduction

In [3], the wave equation with nonlinear damping and source terms, on a bounded domain $\Omega$ of $\mathbb{R}^n$, has been studied. Namely, the authors considered the following problem:

$$\begin{array}{l} u_{tt} - \Delta u + a u_t |u_t|^{m-1} = b u |u|^{p-1}, \qquad x \in \Omega, \qquad t > 0 \\ u(x,t) \;=\; 0, \qquad x \in \partial\Omega, \quad t > 0 \\ u(x,0) \;=\; u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \end{array} \tag{1}$$

where $a, b > 0, p, m > 1$ and $\Omega$ is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. They discussed the interaction between the damping and the source terms and established a global existence, as well as, a blow-up result for the case $a = b = 1$ and $p, m$ satisfying some conditions. This result was generalized to an abstract setup by Levine and Serrin [9], Levine and Park [10] and Vitillaro [11].

It is well known that in the case where $a = 0$, the source term destabilizes the solution and causes a finite time blow-up (See [2],[7],[8]). On the other hand, if $b = 0$, the nonlinear damping term ensures the global existence for small initial data (See [4]).

In [6] and [12], the linear wave equation together with a nonlinear feedback at the boundary was investigated. Especially, the authors studied the following problem:

$$\begin{array}{ll} u_{tt} - \Delta u = 0, & x \in \Omega, \quad t > 0 \\ \partial u \partial \nu(x,t) = -m(x).\nu(x) g(u_t), & x \in \Gamma_0, \quad t > 0 \\ u(x,t) = 0, & x \in \Gamma_1, \quad t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{array} \tag{2}$$

where $m(x) = x - x_0$, $x_0 \in \mathbb{R}^n, \Gamma_0 = \{x \in \partial\Omega : m(x).\nu(x) > 0\}$, and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, with $\Gamma_1 \neq \phi$. They showed that, under certain growth conditions on $g$

and for suitable initial data, the energy of the solution decays exponentially. A similar problem has also been studied by Aliev and Khanmamedov [1] in the $n$-dimensional open unit cube.

In the present paper, we consider the following initial-boundary value problem:

$$u_{tt}(x,t) - \Delta u(x,t) = -mu_t(x,t) + \bigtriangledown\phi(x)\cdot\bigtriangledown u(x,t) \tag{3}$$
$$+ f(u(x,t)), \qquad x\in\Omega,\ t>0$$

$$u(x,t) = 0, \qquad x\in\partial\Omega, \quad t>0 \tag{4}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x\in\Omega, \tag{5}$$

where $m$ is a strictly positive constant, $\phi \in W^{1,\infty}(\Omega)$, and $\Omega$ is a bounded open domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. We shall show that, for suitably chosen initial data, any solution $u$ lying in

$$C\left([0,\ T);\ H^1(\Omega)\right)\cap C^1\left([0,\ T);\ H^0(\Omega)\right),\ T>0, \tag{6}$$

blows up in finite time. To prove our result we will use a lemma by Kalantarov and Ladyzhenskaya [5], which we state without proof.

Lemma. Assume that a positive twice differentiable function $G(t)$ satisfies, for $t\geq 0$,

$$G(t)G''(t) - (1+\gamma)G'^2(t) \geq -2C_1G(t)G'(t) - C_2G^2(t), \tag{7}$$

where $\gamma>0$ and $C_1, C_2\geq 0$. Then,

a) If $G(0)>0$, $G'(0)+\gamma_2\gamma^{-1}G(0)>0$, and $C_1+C_2>0$ we have $G(t)\to\infty$ as

$$t\to t_m \leq t^* = \frac{1}{2\sqrt{C_1^2+\gamma C_2}}\ln\left(\frac{\gamma_1 G(0)+\gamma G'(0)}{\gamma_2 G(0)+\gamma G'(0)}\right),$$

where $\gamma_1 = -C_1+\sqrt{C_1^2+\gamma C_2}$ $\quad\gamma_2 = -C_1-\sqrt{C_1^2+\gamma C_2}$.

b) If $G(0)>0$, $G'(0)>0$, and $C_1=C_2=0$ we have $G(t)\to\infty$ as

$$t\to t_m\leq t^* = \frac{G(0)}{\gamma G'(0)}.$$

## 2. Blow Up

In order to state our main result, we make the following hypotheses:

$$E_0 := \frac{1}{2}\int_\Omega e^{\phi(x)}[u_1^2(x) + |\bigtriangledown u_0(x)|^2] - F(u_0(x))]dx < 0, \tag{8}$$

$$\int_\Omega e^{\phi(x)}u_0(x)u_1(x)dx \geq \frac{m}{\gamma}\int_\Omega e^{\phi(x)}u_0^2(x)dx, \tag{9}$$

where $F(s) = \int_0^s f(\zeta)d\zeta$ satisfying, for some $\alpha>0$,

$$f(s)s \geq (2+\alpha)F(s),\ \forall\ s\in\mathbb{R}. \tag{10}$$

Theorem. Assume that, for $0<\gamma<\alpha/4$, (8) - (9) hold. Then, any solution of (5), satisfying (6), blows up in finite time. ð

We set

$$E(t) := \frac{1}{2}\int_\Omega e^{\phi(x)}[u_t^2(x) + | \bigtriangledown u(x,t)|^2 - F(u(x,t))]dx.$$

By multiplying equation (3) by $e^{\phi(x)}u_t(x,t)$ and integrating over $\Omega$, we get

$$E'(t) = -m\int_\Omega e^{\phi(x)}u_t^2(x,t)dx;$$

hence $E(t) \leq E_0 < 0$. We then define

$$G(t) := \frac{1}{2}\int_\Omega e^{\phi(x)}u^2(x,t)\,dx + \frac{1}{2}\beta(t+t_0)^2$$

and differentiate $G$ twice, to get

$$G'(t) = \int_\Omega e^{\phi(x)}uu_t(x,t)\,dx + \beta(t+t_0).$$

and

$$G''(x,t) = \int_\Omega e^{\phi(x)}[u_t^2(x,t) + uu_{tt}(x,t)]dx + \beta.$$

By using equation (3), we arrive at

$$G''(t) = \beta + \int_\Omega e^{\phi(x)}[u_t^2 - | \ \bigtriangledown u \ |^2 + uf(u)](x,t)\,dx - m\int_\Omega e^{\phi(x)}u_tu(x,t)dx$$

$$= \beta-(\alpha+2)E(t)+(2+\frac{\alpha}{2})\int_\Omega e^{\phi(x)}u_t^2(x,t)\,dx+\frac{\alpha}{2}\int_\Omega e^{\phi(x)}|\bigtriangledown u(x,t)|^2\,dx \quad (11)$$

$$+\int_\Omega e^{\phi(x)}[uf(u) - (2+\alpha)F(u)] - m\int_\Omega e^{\phi(x)}u_tu(x,t)dx.$$

Next we investigate

$$Q(t) := G(t)G''(t) - (\gamma+1)G'^2(t) =$$

$$G(t)\left(\beta - (\alpha+2)E(t) + (2+\frac{\alpha}{2})\int_\Omega e^{\phi(x)}u_t^2(x,t)dx + \frac{\alpha}{2}\int_\Omega e^{\phi(x)}| \bigtriangledown u(x,t)|^2dx\right.$$

$$\left.-m\int_\Omega e^{\phi(x)}u_tu(x,t)dx\right) - (\gamma+1)\left(\int_\Omega e^{\phi(x)}uu_t(x,t)\,dx + \beta(t+t_0)\right)^2.$$

By using Young's inequality and (11) we obtain

$$Q(t) \geq G(t)\left(\beta - (\alpha+2)E(t) + (2+\frac{\alpha}{2})\int_\Omega e^{\phi(x)}u_t^2(x,t)\,dx\right. \quad (12)$$

$$\left.+\frac{\alpha}{2}\int_\Omega e^{\phi(x)}| \bigtriangledown u(x,t)|^2\,dx - m\int_\Omega e^{\phi(x)}u_tu(x,t)dx\right)$$

$$-(\gamma+1)\left((1+\frac{\varepsilon}{2})\{\int_\Omega e^{\phi(x)}uu_t(x,t)\,dx\}^2+(1+\frac{1}{2\varepsilon})\beta^2(t+t_0)^2\right).$$

And by exploiting Schwartz inequality, (12) becomes

$$Q(t)\geq G(t)\left(\beta-(\alpha+2)E(t)+(2+\frac{\alpha}{2})\int_\Omega e^{\phi(x)}u_t^2(x,t)\,dx\right.$$

$$\left.+\frac{\alpha}{2}\int_\Omega e^{\phi(x)}|\bigtriangledown u(x,t)|^2\,dx-m\int_\Omega e^{\phi(x)}u_tu(x,t)dx\right)-(\gamma+1)\left(2(1+\frac{\varepsilon}{2})G(t)\right.$$

$$\left.\times\int_\Omega e^{\phi(x)}u_t^2(x,t)\,dx+2(1+\frac{1}{2\varepsilon})\beta G(t)\right) \tag{13}$$

$$\geq G(t)\left(\beta-(\alpha+2)E(t)+[2+\frac{\alpha}{2}-2(1+\frac{\varepsilon}{2})(\gamma+1)]\int_\Omega e^{\phi(x)}u_t^2(x,t)\,dx\right.$$

$$\left.+\frac{\alpha}{2}\int_\Omega e^{\phi(x)}|\bigtriangledown u(x,t)|^2\,dx-2(1+\frac{1}{2\varepsilon})(\gamma+1)\beta\right)-mG(t)G'(t)$$

$$\geq G(t)\left([\frac{\alpha}{2}-\varepsilon-\gamma(\varepsilon+2)]\int_\Omega e^{\phi(x)}u_t^2(x,t)\,dx+\frac{\alpha}{2}\int_\Omega e^{\phi(x)}|\bigtriangledown u(x,t)|^2\,dx\right.$$

$$\left.-(1+\frac{1}{\varepsilon}+2\gamma+\frac{\gamma}{\varepsilon})\beta-(\alpha+2)E(t)\right)-mG(t)G'(t),\quad\forall\,\varepsilon>0.$$

By choosing $\varepsilon=2(\alpha/4-\gamma)/(\gamma+1)$, we easily see that $[\alpha/2-\varepsilon-\gamma(\varepsilon+2)]=0$; hence (13) reduces to

$$Q(t)\geq G(t)\left(-(1+\frac{1}{\varepsilon}+2\gamma+\frac{\gamma}{\varepsilon})\beta-(\alpha+2)E(t)\right)-mG(t)G'(t). \tag{14}$$

We then choose $\beta>0$ small enough so that $-(1+\frac{1}{\varepsilon}+2\gamma+\frac{\gamma}{\varepsilon})\beta-(\alpha+2)E(t)\geq 0$, consequently (14) yields

$$G(t)G''(t)-(\gamma+1)G'^2(t)\geq -mG(t)G'(t). \tag{15}$$

Therefore condition (7) of the lemma is satisfied with $C_1=m/2$ and $C_2=0$. It remains to verify condition a). At this point we first note that

$$F(0)=\int_\Omega e^{\phi(x)}u_0^2(x)\,dx+\beta t_0^2>0,\quad\forall\beta,\ t_0>0.$$

Next we consider

$$F'(0)-\frac{m}{\gamma}F(0)=\int_\Omega e^{\phi(x)}u_0u_1(x)\,dx+\beta t_0-\frac{m}{\gamma}\int_\Omega e^{\phi(x)}u_0^2(x)\,dx-\frac{m}{\gamma}\beta t_0^2$$

$$=\int_\Omega e^{\phi(x)}u_0u_1(x)\,dx-\frac{m}{\gamma}\int_\Omega e^{\phi(x)}u_0^2(x)\,dx+\beta t_0(1-\frac{m}{\gamma}t_0).$$

By using (9) and choosing $t_0<\gamma/m$, all conditions of the lemma are satisfied. Therefore, $G(t)$ blows up in finite time $t_m$.

**Remark 3.1**. Note that no assumption has been made on the size of the initial data. In fact, the blow up takes place even for small data provided that (8) and (9) are satisfied.

**Remark 3.2.** For $m = 0$, one can prove a similar result assuming either (8) only, or $E_0 \leq 0$ and $\int_\Omega e^{\phi(x)} u_0 u_1(x)\, dx > 0$.

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