1 The Linear case

Our aim is to solve

$$\begin{cases} \frac{du}{dt} + Au = 0, \quad \forall t \in \quad [0, +\infty) \\ u(0) = u_0. \end{cases} \tag{.1}$$

Definition 1.1: Let H be a Hilbert space and $A : D(A) \subset H \longrightarrow H$ be a non bounded linear operator. We say that A is monotone, if

$$(Av, v) \geq 0, \quad \forall v \in D(A).$$

We say that A is maximal monotone, if, in addition, $R(I + A) = H$; i.e.

$$\forall f \in H, \quad \exists u \in D(A) \text{ such that } u + Au = f.$$

Proposition 1.1: *Let A be a maximal monotone operator. Then,*

(a) $D(A)$ *is dense in* H.

(b) A *is closed.*

(c) *For any* $\lambda > 0, \quad I + \lambda A : D(A) \to H$ *is bijective with* $(I + \lambda A)^{-1}$ *bounded and*

$$\|(I + \lambda A)^{-1}\|_{\ell(H)} \leq 1.$$

Proof.

(a) Since A is maximal monotone. So for $f \in H, \ \exists v_0 \in D(A)$ such that $v_0 + Av_0 = f$.

If $(f, v) = 0, \ \forall v \in D(A)$ then

$$0 = (f, v_0) = ||v_0||^2 + (Av_0, v_0).$$

Thus $||v_0||^2 = 0$ since $(Av_0, v_0) \geq 0 \Rightarrow v_0 = 0 \Longrightarrow f = 0$. Therefore $D(A)$ is dense in H.

(b) It is easy to verify that

$$u + Au + f$$

has a unique solution. Because if $\bar{u}$ is another solution then

$$u - \bar{u} + A(u - \bar{u}) = 0.$$

This gives

$$||u - \bar{u}||^2 + (A(u - \bar{u}), u - \bar{u}) = 0$$

$\Rightarrow ||u - \bar{u}||^2 = 0 \Rightarrow u = \bar{u}$. Also,

$$||u\|^2 + (Au, u) = (f, u) \Rightarrow ||u||^2 \leq (f, u) \Rightarrow ||u|| \leq ||f||.$$

So the inverse operator $(I + A)^{-1}$ is bounded from H to $D(A)$ with

$$\|(I + A)^{-1}\| \leq 1.$$

Let (u_n) be a sequence in $D(A)$ such that $u_n \to u$ and $Au_n \to g$. We have $u_n + Au_n \to u + g$; so,

$$u_n = (I+A)^{-1}(u_n+Au_n) \to (I+A)^{-1}(u+g) \in D(A).$$

Thus

$$u = (I + A)^{-1}(u + g) \in D(A).$$

So,

$$(I + A)u = u + g \Leftrightarrow Au = g.$$

We conclude that A is closed.

(c) Suppose that there exists λ_0 such that $R(I+\lambda_0 A) = H$. We will show that

$$\forall \lambda > \frac{\lambda_0}{2}, \qquad R(I+\lambda A) = H.$$

In this case, as in (b), we have for every $f \in H$, there exists a unique $u \in D(A)$ such that $u + \lambda_0 Au = f$ and the "inverse" operator $(I+\lambda_0 A)^{-1}$ is bounded with $\|(I+\lambda_0 A)^{-1}\| \leq 1$. We consider, now,

$$u + \lambda Au = f, \quad \lambda > 0 \tag{.2}$$

Equation (2) can be written as:

$$u + \lambda_0 Au = \frac{\lambda_0}{\lambda} f + \left(1 - \frac{\lambda_0}{\lambda}\right) u \tag{.3}$$

For $\lambda > \frac{\lambda_0}{2}$, it is easy to see that $\left|1 - \frac{\lambda_0}{\lambda}\right| < 1$. By using the Banach fixed point theorem, we easily prove that (3) has a unique solution.

We conclude then, if A is maximal monotone then $I + A$ is onto. So, by the previous analysis, $I + \lambda A$ is onto for any $\lambda > \frac{1}{2}$. So, for any $\lambda > \frac{1}{4}, \ldots$ and so on. Thus $(I + \lambda A)$ is onto for any $\lambda > 0$.

Theorem 1.2. (Hille-Yosida) *Let A be a maximal monotone operator in a Hilbert space H. Then, for any $u_0 \in D(A)$, there exists a unique function*

$$u \in C^1((0,+\infty); H) \cap C([0,+\infty); D(A)),$$

such that

$$\begin{cases} \frac{du}{dt} + Au = 0, \quad \forall t \in \quad [0, +\infty) \\ u(0) = u_0. \end{cases}$$

Moreover, we have

$$||u(t)|| \leq ||u_0||, \quad \left\| \frac{du}{dt}(t) \right\| = \|Au(t)\| \leq \|Au_0\|, \forall t \geq 0.$$

Idea of proof.

a) **Uniqueness:** Let u and $\bar{u}$ be two solutions. So,

$$\left(\frac{d}{dt}(u - \bar{u}), u - \bar{u} \right) = -\left(A(u - \bar{u}), u - \bar{u} \right) \leq 0;$$

that is

$$\frac{1}{2} \frac{d}{dt} \|u(t) - \bar{u}(t)\|^2 \leq 0.$$

Therefore

$$\|u(t) - \bar{u}(t)\|^2 \leq \|u(0) - \bar{u}(0)\|^2 = 0.$$

b) **Existence:** To prove the existence, we treat a more regular problem with the help with Yosida regularizing and Picard method. Then we pass to the limit (see [] p.105-109).

c) **Estimates:** From the equation, we infer that

$$\left(\frac{du}{dt}, u \right) = -(Au, u) \leq 0.$$

That is,

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 \leq 0.$$

Therefore,

$$\|u(t)\|^2 \leq \|u(0)\|^2 = \|u_0\|^2.$$

Also, we have

$$\left(\frac{du}{dt}, \frac{du}{dt}\right) + \left(Au, \frac{du}{dt}\right) = 0.$$

That is,

$$\frac{1}{2}\frac{d}{dt}||Au||^2 = -\left\|\frac{du}{dt}\right\|^2 \leq 0$$

Thus,

$$\|Au\|^2 \leq \|Au(0)\|^2 = \|Au_0\|^2.$$

From the equation, we obtain

$$\left\|\frac{du}{dt}\right\|^2 = \|Au(t)\|^2 \leq \|Au_0\|^2, \quad \forall t \geq 0.$$

Definition 1.2. Given an operator $A : D(A) \subset H \to H$, with $\overline{D(A)} = H$. If we identify $H' = H$ we define the adjoint operator A^* by

$$(Au, v) = (u, A^*v), \ \forall u, v \in D(A).$$

Definition 1.3. A is symmetric, if

$$(Au, v) = (u, Av), \ \forall u, v, \in D(A).$$

A is self-adjoint if

$$A^* = A \text{ and } D(A) = D(A^*).$$

Remark 1.1. If A is bounded then there is no difference between symmetric and selfadjoint. However, any selfadjoint operator is symmetric.

Proposition 1.3. *Let A be a maximum monotone operator. If A is symmetric then it is selfadjoint.*

Proof. Let $J = (I + A)^{-1}$. We verify that J is symmetric. Set $u_1 = Ju$ and $v_1 = Jv$; so that

$$u_1 + Au_1 = u, \quad v_1 + Av_1 = v$$

Since $(u_1, Av_1) = (Au_1, v_1)$, by symmetry of A, then

$$(u_1, v) = (u, v_1); \quad \text{that is} \quad (Ju, v) = (u, Jv).$$

Therefore J is symmetric, hence selfadjoint since $J \in \mathcal{L}(H)$. This implies $(I + A)$ is self-adjoint. So $A = A^*$. To show that $D(A^*) = D(A)$, let $u \in D(A^*)$ and set $f = u + A^*u$

We have

$$(f, v) = (u, v + Av), \ \ \forall v \in D(A);$$

i.e.

$$(f, Jw) = (u, w), \ \ \forall w \in H.$$

Since A is maximum. So,

$$(Jf, w) = (u, w), \forall w \in H.$$

Since J is self-adjoint. This gives $u = Jf \epsilon D(A)$; hence $D(A^*) = D(A)$.

Remark 1.2. A is maximal monotone iff A^* is maximal monotone iff A is densely closed and A and A^* are monotone.

Theorem 1.4 (Existence)

Let A be a maximal monotone self-adjoint operator. Then for each $u_0 \in H$, there exists a function $u \epsilon C([0, +\infty]; H) \cap C^1((0, +\infty); H) \cap C((0, +\infty]; D(A))$

unique satisfying (1). Moreover, we have

$$\|u(t)\| \leq \|u_0\|, \|\frac{du}{dt}\| = \|A(t)\| \leq \frac{1}{t}\|u_0\|, \forall t > 0$$

Remark 1.3. Notice here that the derivatives may blow-up at zero.

1.1 Nonautonomous operators

Let H and V be two Hilbert spaces such that $V \subset H$ with continuous embedding. We identify H with its dual so that we obtain

$$V \subset H \subset V'.$$

Suppose that for any $t \in [0,T], T > 0$, the bilinear form

$$a(t;.,.) : V \times V \to R$$

satisfies

(i) the function $t \to a(t;u,v)$ is measurable, $\forall u, v \in V$

(ii) $|a(t;u,v)| \leq M\|u\|_V\|v\|_V$ for a.e. $t \in [0,T], \forall u, v \in V$

(iii) $a(t;u,u) \leq \alpha\|u\|_V^2 - C\|u\|_H$, for a.e. $\in [0,T], \forall u \in V$.

where $\alpha > 0$ and C are constants.

Theorem 1.5 (J.L. Lions)

Given $f \in L^2(0,T;V')$ and $u_0 \in H$. Suppose that a bilinear form $a(t;.,.)$ on V satisfies $(i)-$

(iii). *Then there exists a unique function*

$u \in L^2(0,T;V)\cap C([0,T];H), \quad \frac{du}{dt} \in L^2(0,T;V')$

such that, for for a.e. $t \in [0,T], \forall v \in V$

$$\langle \frac{du}{dt}(t), v\rangle + a(t;u(t),v) = \langle f(t), v\rangle,$$
$$u(0) = u_0.$$

Proof. (see Lions-Magenes)

Application 1.2.

Given the heat equation with a history term

$$\begin{cases} u_t - \Delta u + \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau = 0, & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \ x \in \partial\Omega\,, t \geq 0 \\ u(x,0) = u_0(x), \end{cases} \tag{.4}$$

for $\Omega \subset \mathbb{R}^n$ a bounded domain with a smooth boundary Γ.

Theorem 1.6: *Suppose that* g is continuous with

$$1 - \int_0^{+\infty} g(s)ds = l > 0$$

Then, for any $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$*, there exists a unique function*

$$u \in L^2(0,T;H_0^1(\Omega))$$

satisfying, for $a.e\ t \in [0,T], \forall v \in H_0^1(\Omega)$,

$$\int_\Omega \frac{du(x,t)}{dt} v(x)dx + \int_\Omega \nabla u(x,t).\nabla v(x)dx$$

$$-\int_\Omega\int_0^t g(t-s)\nabla u(x,s).\nabla v(x)dsdx = \int_\Omega f(x,t)v(x)dx$$
$$u(0) = u_0.$$

Proof.

Define the bilinear form
$$a(t,.,.):\ H_0^1(\Omega)\times H_0^1(\Omega)\to \mathbb{R}$$
by
$$\begin{aligned} a(t,u,v) &= \int_\Omega \nabla u(x).\nabla v(x)dx \\ &\quad - \int_\Omega\int_0^t g(t-s)\nabla u(x).\nabla v(x)dsdx \\ &= (1-\int_0^t g(s)ds)\int_\Omega \nabla u(x).\nabla v(x)dx \end{aligned}$$
We verify the hypotheses $(i)-(iii)$ of Theorem 1.5
(i) $|a(t_1,u,v)-a(t_2,u,v)| \leq ||u||_V||v||_V|\int_{t_1}^{t_2} g(s)ds|$. Thus $a(.,u,v)$ is absolutely continuous hence measurable for all $t\geq 0$ and $u,v\in H_0^1(\Omega)$
(ii) $|a(t,u,v)| \leq ||u||_V||v||_V$, or all $t\geq 0$ and $u,v\in H_0^1(\Omega)$
(iii) $a(t,u,u)\geq l||u||_V^2$.

All conditions are verified. Therefore there exists a unique weak solution for problem (4).

2 Nonlinear Case

Suppose that V is a Banach space and H is a Hilbert space such that $V \subset H \subset V'$ with continuous embedding and V is dense in H.

Definition 2.1. We say that $A : V \to V'$ is montone if

$$\langle A(u) - A(v), \quad u - v \rangle \geq 0 \quad \forall u, v \in V.$$

If, in addition, A satisfies

$$\langle A(u) - A(v), \quad u - v \rangle > 0 \quad \forall u \neq v \in V$$

it is called strictly monotone.

Definition 2,2. We say that $A : V \to V'$ is hemi-continuous if we have, $\forall u, v, w \in V$, the function $\lambda \to \langle A(u + \lambda v), w \rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$.

Definition 2.3. Let V be a Banach space. $A : V \to V'$ is bounded if for any bounded set $S \subset V$, $A(S)$ is bounded in V'.

Given $f \in L^{p'}(0, T; V')$ and $u_0 \in H$. We would like to find a function $u : [0, T] \longrightarrow V$ such that

$$\begin{cases} \frac{du(t)}{dt} + A(u((t)) = f(t) \text{ in } V' \\ u(0) = u_0 \in H \end{cases} \tag{.5}$$

Remark 2.1: Suppose that $u \in L^p(0, T; V)$ such that $u' \in L^{p'}(0, T; V')$. Then $u : [0, T] \longrightarrow H$ is continuous i.e. $u \in C([0, T]; H)$.

Theorem 2.1: *Let V and H be such as in above with V separable. Let $A : V \longrightarrow V'$ be (nonlinear) operator satisfying:*

(1) $A : V \rightarrow V'$ *is hemicontinuous with* $||A(v)||_{V'} \leq C||v||_V^{p-1}$

(2) A *is monotone.*

(3) $\langle A(v), v\rangle \geq \alpha ||v||_V^p, \quad \alpha > 0, \quad \forall v \in V, \quad (1 < p < +\infty).$

Then, given $f \in L^{p'}(0, T; V')$ *and* $u_0 \in H$, *problem (5) has a unique solution* $u \in L^p(0, T; V)$.

Proof. P.156-163 (lions).

Remark 2.2: The above remark implies that $u \in C([0, T]; H)$ so $u(0) = u_0$ has meaning.

Example 2.1: We consider, for $p > 1$, the problem

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f \text{ in } \Omega$$

$$u = 0 \quad \text{on} \quad \sum = \Gamma\times]0, t[\text{ (.6)}$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega$$

for $\Omega \subset \mathbb{R}^n$ a bounded domain with a smooth boundary Γ.

Theorem 2.2: *Suppose that*

$$f \in L^{p'}(0, T; W^{-1,p'}(\Omega)), \quad \frac{1}{p} + \frac{1}{p'} = 1$$

and $u_0 \in L^2(\Omega)$. *Then, there exists a unique func-*

tion

$$u \in L^p(0,T;W_0^{1,p}(\Omega))$$

satisfying (6).

Proof: We verify the conditions of the previous theorem. Let $H = L^2(\Omega)$ and $V = W_0^{1,p}(\Omega)$, So $V' = W^{-1,p'}(\Omega)$. We know from previous considerations that the operator

$$A : V \longrightarrow V'$$

$$v \longrightarrow A(v) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i}\right)$$

is “strictly” monotone and hemicontinuous. Also,

$$\langle A(v), v\rangle = \int_\Omega \sum_{i=1}^{n} \left|\frac{\partial v}{\partial x_i}\right|^p = ||\nabla v||_p^p, \forall v \in V, 1 < p < \infty.$$

Consequently

$$||A(v)||_{V"} \leq ||\nabla v||_p^{p-1} = ||v||_V^{p-1}.$$

We conclude that (5) has a unique solution

$$u \in L^p(0,T;W_0^{1,p}(\Omega)).$$

3 Decay

We consider, for $p > 1$, the problem

$$\frac{\partial u}{\partial t} - div\left(|\nabla u|^{p-2}\nabla u\right) = 0 \text{ in } \Omega$$
$$u = 0 \quad \text{on} \quad \sum = \Gamma\times]0,t[\qquad (.7)$$
$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega$$

for $\Omega \subset \mathbb{R}^n$ a bounded domain with a smooth boundary Γ.

Theorem 3.1: *Suppose that* $u_0 \in L^2(\Omega)$. *Then, the solution satisfies, for* c *and* $C > 0$,

$$||\nabla u||_p \leq Ce^{-ct}, \quad \forall t \geq 0, \quad p = 2 \qquad (.8)$$

$$||\nabla u||_p \leq C(1+t)^{-2/p(p-2)}, \forall t \geq 0, \ p > 2 \qquad (.9)$$

Proof. Let

$$E(t) = \frac{1}{p}||\nabla u||_p^p \qquad (.10)$$

Multiply (7) by u_t and integrate over Ω :

$$E'(t) = -\int_\Omega u_t^2 dx \leq 0$$

So

$$0 \leq E(t) \leq E(0)$$

define

$$H(t) = E(t) + \frac{1}{2}\int_\Omega u^2 dx \qquad (.11)$$

It is clear

$$\begin{aligned} H(t) &\leq E(t) + \frac{1}{2}C_p^2||\nabla u||_p^2 \\ &\leq E(t) + \frac{1}{2}C_p^2[E(t)]^{2/p} \\ &\leq \left([E(t)]^{1-2/p} + \frac{1}{2}C_p^2\right)[E(t)]^{2/p} \\ &\leq \left([E(0)]^{1-2/p} + \frac{1}{2}C_p^2\right)[E(t)]^{2/p} \\ &= c[E(t)]^{2/p} \end{aligned}$$

Also

$$\begin{aligned} H'(t) &= -\int_\Omega u_t^2 dx - \int_\Omega |\nabla u|^2 dx \\ &\leq -pE(t) \leq -cpH^{p/2}(t) \end{aligned} \qquad (.12)$$

Case 1. $p = 2$

$$H'(t) \leq -cpH(t)$$

Simple integration gives

$$H(t) \leq H(0)e^{-cpt}, \quad t \geq 0.$$

Hence (10), (11) $\Longrightarrow$

$$||\nabla u||_p \leq (pH(0))^{1/p}\, e^{-ct}, \quad t \geq 0.$$

Case 2. $p > 2$

Integrate (12):

$$H(t) \leq \left(cp\frac{p-2}{2}t + H^{(2-p)/2}(0) \right)^{-2/(p-2)}$$
$$\leq k(1+t)^{-2/(p-2)}$$

So (10), (11) $\Longrightarrow$

$$||\nabla u||_p \leq (kp)^{1/p}(1+t)^{-2/p(p-2)}$$