On the existence and nonexistence of solutions of a nonlinear hyperbolic system describing heat propagation by second sound.

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Abstract

In this work, we consider a nonlinear hyperbolic system describing heat propagation, where the heat flux is given by Cattaneo's law. We state the global existence theorem, presented in [4], and establish a blow up result.

Keywords heat, second sound, nonlinear, hyperbolic, global existence, blow-up.
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1 Introduction

In the absence of deformation, heat propagation in one spatial dimension body is governed by the following equation of balance of energy

$$e_t + q_x = 0; \tag{1.1}$$

where the internal energy e and the heat flux q are functions of (x, t) and a subscript denotes a partial derivative with respect to the relevant variable. In Fourier's theory of heat conduction, the internal energy depends on the absolute temperature only; i.e.

$$e = \hat{e}(\theta) \tag{1.2}$$

whereas the heat flux is given by the relation

$$q = -\kappa(\theta)\theta_x: \tag{1.3}$$

As a consequence, the evolution of the heat flux and the absolute temperature is given by the system

$$\begin{aligned} q + \kappa(\theta)\theta_x &= 0 \\ q_x + \hat{e}'(\theta)\theta_t &= 0, \end{aligned} \tag{1.4}$$

where κ and $\hat{e}'$ are strictly positive functions characterizing the material in consideration. In the case where $\hat{e}'$ and κ are independent of θ, we get the familiar linear heat equation

$$\theta_t = k\theta_{xx}, \quad k = \frac{\kappa}{\hat{e}'}. \tag{1.5}$$

This equation provides a useful description of heat conduction under a large range of conditions and predicts an infinite speed of propagation; that is, any thermal disturbance at one point has an instantaneous effect elsewhere in the body. This is not always the case. In fact, experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox (infinite speed propagation) and disturbances which are almost entirely thermal may propagate in a finite speed. This phenomenon in dielectric crystals is called second sound.

These observations go back to 1948, when Cattaneo [2] proposed, in place of (1.3), a new constitutive relation

$$\tau(\theta)q_t + q = -\kappa(\theta)\theta_x, \tag{1.6}$$

where τ and κ are strictly positive functions depending on the absolute temperature. With this relation, the internal energy given by (1.2) is no longer compatible with the second law of thermodynamics. Coleman, Fabrizio, and Owen [3] showed in 1982 that, if (1.6) is adopted then compatibility with thermodynamics requires that (1.2) be replaced by

$$e = \hat{e}(\theta, q) = a(\theta) + b(\theta)q^2, \tag{1.7}$$

where b is a function determined by τ and κ. In particular

$$b(\theta) > 0. \tag{1.8}$$

Thus (1.1), (1.6), and (1.7) combined yield the following system governing the evolution of θ and q

$$\begin{aligned} q_x + (a'(\theta) + b'(\theta)q^2)\theta_t + 2b(\theta)qq_t &= 0 \\ \tau(\theta)q_t + q + \kappa(\theta)\theta_x &= 0. \end{aligned} \tag{1.9}$$

Global existence and decay of classical solutions to the Cauchy problem, as well as to some initial boundary value problems, have been established by Coleman, Hrusa, and Owen [4]. In their paper, the authors used a classical energy argument to prove their result. As they pointed out the method based on the nonlinear semigroup theory, presented in [8] is applicable to their initial value problem. Concerning the formation of singularities, Messaoudi [9] studied the following system

$$\begin{aligned} \tau(\theta)q_t = q + \kappa(\theta)\theta_x &= 0 \\ c(\theta)\theta_t + q_x &= 0 \end{aligned} \tag{1.10}$$

and showed, under the same restrictions on τ, c and κ, that classical solutions to the cauchy problem break down in finite time if the initial data are chosen small in L^1 norm with large enough derivatives.

In this article, we consider a system equivalent to (1.9) and show that, under the same conditions on the initial data, a blow up result can be obtained. This work is divided into two sections. In section two we state, without proof, a global existence result. In section three we establish our main result.

2 Global existence

To derive the equations, we assume that $\tilde{e}$ is a c^1 function, at least, in a neighbourhood V of $(0;0)$ and

$$a'(0) > 0; \tag{2.1}$$

hence

$$\tilde{e}_\theta(0;0) > 0. \tag{2.2}$$

Therefore, we can choose V so that

$$\tilde{e}_\theta(\theta;q) > 0; \quad \forall\, (\theta;q) \in V. \tag{2.3}$$

In this case, θ can be expressed in terms of $(e;q)$; i.e.

$$\theta = \tilde{e}(e;q). \tag{2.4}$$

By combining (1.7) and (2.4), we easily arrive at

$$\theta_x = \frac{e_x - 2b(\tilde{e}(e;q))qq_x}{a'(\tilde{e}(e;q)) + b'(\tilde{e}(e;q))q^2}. \tag{2.5}$$

Thus, by considering (1.1), (1.6), (2.4), (2.5), we get the system of equations governing the evolution of e and q

$$\begin{aligned} \sigma(e;q)q_t + \mu(e;q)q &= -e_x + \lambda(e;q)qq_x \quad (2.6)\\ e_t &= -q_x; \quad x \in \mathbb{R};\ t \geq 0; \quad (2.7)\end{aligned}$$

where

$$\begin{aligned} \sigma(\xi;\eta) &= \frac{\tau(\tilde{e}(\xi;\eta))(a'(\tilde{e}(\xi;\eta)) + b'(\xi;\eta)\eta^2)}{\kappa(\tilde{e}(\xi;\eta))}\\ \mu(\xi;\eta) &= \frac{a'(\tilde{e}(\xi;\eta)) + b'(\tilde{e}(\xi;\eta))\eta^2}{\kappa(\tilde{e}(\xi;\eta))}\\ \lambda(\xi;\eta) &= 2b(\tilde{e}(\xi;\eta)). \end{aligned} \tag{2.8}$$

We, thus, seek classical solutions to the system (2.6), (2.7) which satisfy the initial conditions

$$e(x;0) = e_0(x); \quad q(x;0) = q_0(x); \quad x \in \mathbb{R}. \tag{2.9}$$

Note that, by virtue of the assumptions on τ, κ, a, b, the functions σ, μ remain bounded away from zero in some neighbourhood V of $(0,0)$; i.e.

$$\sigma(\xi, \eta) \geq \underline{\sigma} > 0, \quad \mu(\xi, \eta) \geq \underline{\mu} > 0, \quad \forall (\xi, \eta) \in V. \tag{2.10}$$

Theorem 2.1. Assume that $\sigma, \mu, \geq$ are c^2 functions satisfying (2.10). Then there exists a small positive constant δ such that for any e_0, q_0 in $H^2(\mathbb{R})$ satisfying

$$\|e_0\|_2^2 + \|q_0\|_2^2 < \delta^2, \tag{2.11}$$

the initial value problem (2.6), (2.7), (2.9) possesses a unique global solution (e, q) with

$$e, q \in \bigcap_{i=0}^{2} C^i([0, +\infty); H^{2-i}(\mathbb{R})) \tag{2.12}$$

and

$$e(\cdot, t), e_x(\cdot, t), e_t(\cdot, t), q(\cdot, t), q_x(\cdot, t), q_t(\cdot, t) \to 0 \tag{2.13}$$

in $L^\infty(\mathbb{R})$ and uniformly in $\mathbb{R}$ as $t \to +\infty$.

Remark 2.1. For the proof, we refer the reader to [4].

Remark 2.2. By the Sobolev embedding theorem, the solution

$$e, q \in C^1(\mathbb{R} \times [0, +\infty)); \tag{2.14}$$

hence it is a classical solution.

3 Formation of Singularities

This section is devoted to the statement and the proof of our blow up result. To achieve this goal, we rewrite the problem (2.6),(2.7),(2.9) in the following form

$$\begin{aligned} q_t &= -\varphi(e, q)e_x + 2b(e, q)\varphi(e, q)qq_x - \psi(e, q)q & (3.1)\\ e_t &= -q_x, \quad x \in \mathbb{R}, \ t \geq 0 & (3.2)\\ e(x, 0) &= e_0(x), \ q(x, 0) = q_0(x), \quad x \in \mathbb{R}. & (3.3) \end{aligned}$$

Note that, by virtue of the assumptions on τ, κ, and a, we have

$$\begin{aligned} 0 < \psi(\xi, \eta) \leq \overline{\psi}, \quad \forall (\xi, \eta) \in \mathbb{R}^2 \\ \varphi(\xi, \eta) \geq \underline{\varphi} > 0, \quad \forall (\xi, \eta) \in B. \end{aligned} \tag{3.4}$$

where B is a ball in $\mathbb{R}^2$ centered at $(0,0)$ and with a radius ε_0 to be chosen suitably.

Lemma Assume that φ, b, ψ are C^2 functions satisfying (3.4). Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial data e_0, q_0 in $H^2(\mathbb{R})$ obeying

$$|e_0(x)| < \delta, \quad |q_0(x)| < \delta, \quad \forall x \in \mathbb{R}; \tag{3.5}$$

the solution of (3.1) - (3.3) satisfies

$$|e(x, t)| < \varepsilon, \quad |q(x, t)| < \varepsilon, \quad \forall x \in \mathbb{R}, \ t \geq 0. \tag{3.6}$$

Proof. To carry out the proof, we define

$$
\begin{aligned}
r(x;t) &:= e(x;t) + \int_0^{q(x;t)} \alpha(e(x;t);\xi)\,d\xi \\
s(x;t) &:= e(x;t) + \int_0^{q(x;t)} \beta(e(x;t);\xi)\,d\xi
\end{aligned}
\tag{3.7}
$$

where α and β are C^1 functions satisfying the linear problems :

$$
\begin{aligned}
\alpha_y(y;z) - \gamma(y;z)\alpha_z(y;z) &= \gamma_z(y;z)\alpha(y;z) \\
\alpha(y;0) &= \frac{1}{\sqrt{\varphi(y;0)}} > 0; \quad (y;z) \in B
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
\beta_y(y;z) + \rho(y;z)\beta_z(y;z) &= -\rho_z(y;z)\beta(y;z) \\
\beta(y;0) &= \frac{1}{\sqrt{\varphi(y;0)}} > 0; \quad (y;z) \in B;
\end{aligned}
\tag{3.9}
$$

where

$$
\begin{aligned}
\rho &:= \sqrt{\varphi + (b\varphi q)^2} - b\varphi q \\
\gamma &:= \sqrt{\varphi + (b\varphi q)^2} + b\varphi q.
\end{aligned}
\tag{3.10}
$$

The problems (3.8), (3.9) are first order linear. The solution can be obtained, at least, in a neighbourhood of (0,0) by using the classical method of characteristics (see e.g. [1], [5]). Also by using (3.4), we can choose $\varepsilon_0 <$" such that

$$
\begin{aligned}
0 < \underline{\alpha} \le \alpha(y;z) \le \bar{\alpha} \\
0 < \underline{\alpha} \le \beta(y;z) \le \bar{\alpha}
\end{aligned}
\tag{3.11}
$$

and

$$
\rho(y;z) \ge \bar{\rho} > 0; \quad \gamma(y;z) \ge \bar{\rho}; \tag{3.12}
$$

for any $(y;z) \in B$. We then introduce the differential operators

$$
\begin{aligned}
\partial_t^+ &:= \frac{\partial}{\partial t} + \rho(e;q)\frac{\partial}{\partial x} \\
\partial_t^- &:= \frac{\partial}{\partial t} - \gamma(e;q)\frac{\partial}{\partial x}.
\end{aligned}
\tag{3.13}
$$

By noting that $\alpha;\beta$ satisfy the integral equations

$$
\begin{aligned}
1 + \int_0^z \alpha_y(y;\xi)\,d\xi &= \gamma(y;z)\alpha(y;z) \\
1 - \int_0^z \beta_y(y;\xi)\,d\xi &= \rho(y;z)\beta(y;z);
\end{aligned}
\tag{3.14}
$$

direct calculations then yield

$$\partial_t^+ r = -\psi(e;q)q; \tag{3.15}$$

$$\partial_t^- s = -\psi(e;q)q. \tag{3.16}$$

To this end, we define the nonnegative Lipshitz functions

$$R(t) := \max_x |r(x,t)|, \quad S(t) := \max_x |s(x,t)|, \quad t \in [0,T). \tag{3.17}$$

The maxima in (3.17) are attained since r and s die at infinity. Thus for any $t \in [0,T)$, there exist $\hat{x}, x \in \mathbb{R}$ such that

$$\begin{aligned} R(t) &= |r(\hat{x},t)|, & (3.18)\\ S(t) &= |s(x,t)|. & (3.19) \end{aligned}$$

Also by the definition of R and S, we have

$$\begin{aligned} R(t-h) &\geq |r(\hat{x} - \rho(e(\hat{x},t), q(\hat{x},t))h, t-h)|, & (3.20)\\ S(t-h) &\geq |r(x + h\gamma(e(x,t), q(x,t)), t-h)|, & (3.21) \end{aligned}$$

for any $h \in (0,t)$, hence by subtracting (3.20) from (3.18) and (3.21) from (3.19), dividing by h, and letting h go to zero, we get

$$\begin{aligned} \dot{R}(t) &\leq |\partial_t^+ r(\hat{x},t)| \leq \bar{\psi}|q(\hat{x},t)|,\\ \dot{S}(t) &\leq |\partial_t^- r(x,t)| \leq \bar{\psi}|q(x,t)|, \end{aligned} \tag{3.22}$$

for almost each t in $[0,T)$. We then use (3.7) and (3.12) to arrive at

$$|q(x,t)| \leq \frac{1}{\underline{\alpha}}[R(t) + S(t)] \tag{3.23}$$

whenever (e,q) remains in B. Therefore, combining (3.22) and (3.23), we obtain

$$\frac{d}{dt}(R(t) + S(t)) \leq k(R(t) + S(t)); \quad k = \frac{2\bar{\psi}}{\underline{\alpha}} \tag{3.24}$$

for almost each t and whenever $(e,q) \in B$. A straightforward integration, using Gronwall's inequality, leads to

$$(R(t) + S(t)) \leq (R(0) + S(0))e^{kT} \tag{3.25}$$

for any t, provided that $(e,q) \in B$. We now use (3.7) to majorize e and q as follows

$$\begin{aligned} |q(x,t)| &\leq \frac{R(t) + S(t)}{2\underline{\alpha}}\\ |e(x,t)| &\leq \frac{(2\underline{\alpha} + \bar{\alpha})R(t) + \bar{\alpha}S(t)}{2\underline{\alpha}}; \end{aligned} \tag{3.26}$$

hence, by virtue of (3.25) and (3.26), we have

$$\begin{aligned} |q(x,t)| &\leq \frac{(R(0) + S(0))e^{kT}}{2\underline{\alpha}}\\ |e(x,t)| &\leq \frac{(2\underline{\alpha} + \bar{\alpha})(R(0) + S(0))e^{kT}}{2\underline{\alpha}}; \end{aligned} \tag{3.27}$$

whenever $(e(x,t), q(x,t)) \in B$. We then choose $\delta > 0$ so that

$$\frac{(1 + 2\underline{\alpha} + \bar{\alpha})(R(0) + S(0))e^{kT}}{2\underline{\alpha}} < \frac{\varepsilon_0}{2} \tag{3.28}$$

Therefore we conclude, from (3.27) and (3.28), that if $(e, q) \in B$ (i.e. $|e| < \varepsilon_0$, $|q| < \varepsilon_0$) then (e, q) satisfies, in fact,

$$|e(x,t)| < \frac{\varepsilon_0}{2}; \quad |q(x,t)| < \frac{\varepsilon_0}{2}. \tag{3.29}$$

Consequently we arrive, by continuity, at

$$|e(x,t)| < \varepsilon_0 < \varepsilon; \quad |q(x,t)| < \varepsilon_0 < \varepsilon; \quad \forall\, t \in [0, T). \tag{3.30}$$

This completes the proof of the lemma.

Theorem 3.1. Let $\varphi, b,$ and ψ be as in the lemma. Assume further that

$$\varphi_e(0,0) > \left| \sqrt{\varphi(0,0)}\, \varphi_q(0,0) - 2b(0,0)\varphi^2(0,0) \right|. \tag{3.31}$$

Then we can choose initial data $e_0, q_0 \in H^2(\mathbb{R})$ such that the derivatives of the solution (e, q) blow up in finite time.

Remark 3.1. If $b \equiv 0$ and φ is depending on e only then the problem (3.1) – (3.3), as well as the hypothesis (3.31), are reduced to the problem (2.15) of [9].

Proof. We take an x-partial derivative of (3.15) to get

$$(\partial_t^+ r)_x = \partial_t^+ r_x + \rho_x r_x = -(\psi(e,q)q)_x, \tag{3.32}$$

which implies

$$\begin{aligned} \partial_t^+ r_x &= -\rho_x r_x - (\psi(e,q)q)_x \\ &= -(\rho_e e_x + \rho_q q_x) r_x - (\psi(e,q)q)_x. \end{aligned} \tag{3.33}$$

We then use

$$\begin{aligned} e_x &= \frac{\beta r_x + \alpha s_x}{\alpha\beta(\gamma + \rho)} \\ q_x &= \frac{\rho\gamma r_x - \alpha\gamma s_x}{\alpha\beta(\gamma + \rho)} \end{aligned} \tag{3.34}$$

to obtain

$$\partial_t^+ r_x = -\frac{\rho_e + \rho\rho_q}{\alpha(\rho + \gamma)} r_x^2 - \frac{\rho_e - \gamma\rho_q}{\beta(\gamma + \rho)} s_x r_x - (\psi(e,q)q)_x. \tag{3.35}$$

We now set

$$w := H r_x, \tag{3.36}$$

where H is a C^1 solution, at least in a neighbourhood of (0,0), of the linear problem

$$\begin{aligned} H_y(y,z) + \gamma(y,z)H_z &= \frac{\rho_y - \gamma\rho_z}{\rho + \gamma} H(y,z) \\ H(y,0) &= \sqrt{\varphi(y,0)}. \end{aligned} \tag{3.37}$$

By substituting in (3.35) we get

$$\partial_t^+ w = -\frac{\rho_e + \rho\rho_q}{\alpha H(\rho+\gamma)} w^2 - \frac{q\psi H_q}{H} w - H(\psi(e;q)q)_x. \qquad (3.38)$$

By letting $u = -w$, (3.38) becomes

$$\partial_t^+ u = \frac{\rho_e + \rho\rho_q}{\alpha H(\rho+\gamma)} u^2 - \frac{q\psi H_q}{H} u + H(\psi(e;q)q)_x. \qquad (3.39)$$

We also take an x-partial derivative of (3.16) and use (3.34) to obtain, by similar computations,

$$\partial_t^- s_x = \frac{\gamma_e - \gamma\gamma_q}{\beta(\rho+\gamma)} s_x^2 + \frac{\gamma_e + \rho\gamma_q}{\alpha(\gamma+\rho)} r_x s_x - (\psi(e;q)q)_x. \qquad (3.40)$$

We also set

$$v := M s_x; \qquad (3.41)$$

where M is a C^1 solution, at least in a neighbourhood of (0,0), of the linear problem

$$\begin{aligned} M_y(y;z) + \rho(y;z)M_z(y;z) &= \frac{\gamma_y + \rho\gamma_z}{\gamma+\rho} M(y;z) \\ M(y;0) &= \sqrt{\varphi(y;0)}. \end{aligned} \qquad (3.42)$$

By substituting in (3.40), we get

$$\partial_t^- v = \frac{\gamma_e - \gamma\gamma_q}{M\beta(\gamma+\rho)} v^2 + \frac{q\psi M_q}{M} v - M(\psi(e;q)q)_x. \qquad (3.43)$$

We note that the last terms in (3.39) and (3.43) involve only a 'linear' combination of e_x and q_x which can be expressed in terms of u and v. Therefore (3.39) and (3.43) take the forms

$$\partial_t^+ u = \frac{\rho_e + \rho\rho_q}{\alpha H(\rho+\gamma)} u^2 + F_1(e;q)u + F_2(e;q)v \qquad (3.44)$$

$$\partial_t^- u = \frac{\gamma_e - \gamma\gamma_q}{\beta M(\rho+\gamma)} v^2 + G_1(e;q)u + G_2(e;q)v. \qquad (3.45)$$

As in [6] and [7], we define the nonnegative functions

$$\begin{aligned} U(t) &:= \max\left\{\max_x u(x;t); 0\right\} \\ V(t) &:= \max\left\{\max_x v(x;t); 0\right\}. \end{aligned} \qquad (3.46)$$

We fix $t > 0$ with $U(t) > 0$ and/or $V(t) > 0$ and choose $\hat{x}; x \in \mathbb{R}$ such that

$$U(t) = u(\hat{x};t) \quad \text{and/or} \quad V(t) = v(x;t). \qquad (3.47)$$

For every $h \in (0, T - t)$ we have

$$
\begin{aligned}
U(t+h) &\geq u(\hat{x} + h\rho(e(\hat{x},t), q(\hat{x},t)),\ t+h) \\
V(t+h) &\geq u(x - h\gamma(e(x,t), q(x,t)),\ t+h).
\end{aligned} \tag{3.48}
$$

We subtract (3.47) from (3.48), divide by h, and let h go to zero to obtain

$$
D^+U(t) \geq \partial_t^+ u(\hat{x}, t) \quad \text{and/or} \quad D^+V(t) \geq \partial_t^- v(x,t). \tag{3.49}
$$

By noting that

$$
\begin{aligned}
\left(\rho_e + \rho\rho_q\right)(0,0) &= \frac{(\varphi_e + \varphi\varphi_q - 2b\varphi^2)(0,0)}{2\sqrt{\varphi}(0,0)} \\
\left(\gamma_e + \gamma\gamma_q\right)(0,0) &= \frac{(\varphi_e - \varphi\varphi_q + 2b\varphi^2)(0,0)}{2\sqrt{\varphi}(0,0)}
\end{aligned} \tag{3.50}
$$

and by virtue of (3.31), we can choose δ so small that H and M remain strictly positive and

$$
\frac{\gamma_e - \gamma\gamma_q}{M\beta(\gamma + \rho)} \geq 2m; \qquad \frac{\rho_e + \rho\rho_q}{H\alpha(\rho + \gamma)} \geq 2m; \tag{3.51}
$$

for m a positive constant. Therefore, combining all inequalities above, we arrive at the estimate

$$
D^+(U(t) + V(t)) \geq 2m\left(U^2(t) + V^2(t)\right) - K(U(t) + V(t)); \tag{3.52}
$$

where K is an upper bound for $|F_1| + |G_1|$ and $|F_2| + |G_2|$. By setting

$$
W := U - V \tag{3.53}
$$

the estimate (3.52) takes the form

$$
\frac{d}{dt}W(t) \geq mW^2(t) - KW(t) \tag{3.54}
$$

for almost every t in the interval of existence of the solution. We choose initial data small enough in L^1 norm with derivatives such that

$$
\begin{aligned}
u(x,0) &= -\alpha_0 H_0 (q_0' + \gamma_0 e_0')(x) \\
v(x,0) &= -\beta_0 M_0 (q_0' - \rho_0 e_0')(x)
\end{aligned} \tag{3.55}
$$

are large enough to make W in (3.54) blow up in finite time.

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