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DECAY AND GRADIENT ESTIMATE FOR SOLUTIONS OF A QUASILINEAR HEAT EQUATION

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ABSTRACT. In this work we consider an initial boundary value problem related to the equation

$$u_t - div(|\nabla u|^{\alpha-2}\nabla u) = b|u|^{p-2}u, \quad p > \alpha \geq 2 \, , \, b > 0.$$

We give, under suitable conditions on the initial data, a precise estimate on the gradient and prove that the energy of weak solutions decay exponentially for $\alpha = 2$ and in a polynomial rate for $\alpha > 2$ as $t \to \infty$.

1. INTRODUCTION

Research of global existence and finite time blow up of solutions for the initial boundary value problem

$$\begin{aligned} u_t - div(|\nabla u|^{\alpha-2}\nabla u) + f(u) = 0, \qquad x \in \Omega, \qquad t > 0 \\ u(x,t) = 0, \quad x \in \partial\Omega \, , \, t \geq 0 \\ u(x,0) = u_0(x), \qquad x \in \Omega, \end{aligned} \tag{1.1}$$

where $\alpha \geq 2$ and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$, has attracted a great deal of people. The obtained results show that global existence and nonexistence depend roughly on α, the degree of nonlinearity in f, the dimension n, and the size of the initial data. In the early 70's, Levine [6] introduced the concavity method and showed that solutions with negative energy blow up in finite time. Later, this method had been improved by Kalantarov and Ladyzhenskaya [5] to accommodate more situations. This type of results have been extensively generalized and improved by Levine, Park, and Serrin in [7]. The authors, in these papers,

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proved several global and nonglobal existence theorems. On the other hand if f has at most a linear growth then we can find global solutions.

Concerning the asymptotic behavior, Engler, Kawohl, and Luckhaus [2] considered (1.1), with $\alpha = 2$ and showed that for, $f(0) = 0$, $f'(u) \geq a > 0$, and sufficiently small initial datum u_0, the solution satisfies a gradient estimate of the type $||\nabla u||_p \leq Ce^{-\delta t}||\nabla u_0||_p$. This result was also established, under certain geometric conditions on $\partial\Omega$, for an initial boundary problem for the quasilinear equation of the form

$$u_t - div(\sigma(|\nabla u|^2)\nabla u) + f(u) = 0. \tag{1.2}$$

Similar results concerning global existence and gradient estimates have been proved by Nakao and Ohara [8] and Nakao and Chen [9]. It is also worth mentioning that Pucci and Serrin [10] discussed the stability of the rest state for a quasilinear heat system of the form

$$A(t)|u_t|^{m-2}u_t = \Delta u - f(x, u),$$

for $m > 1$ and the source satisfying $(f(x, u), u) \geq 0$. They established a global result of solutions and showed that these solutions tend to the rest state as $t \to \infty$, however no rate of decay has been given.

In this work we consider

$$\begin{array}{c} u_t - div(|\nabla u|^{\alpha-2}\nabla u) = bu|u|^{p-2}, \qquad x \in \Omega, \qquad t > 0 \\ u(x,t) = 0, \quad x \in \partial\Omega\ , t \geq 0 \\ u(x,0) = u_0(x), \qquad x \in \Omega, \end{array} \tag{1.3}$$

$p > \alpha \geq 2$ and show that for suitably chosen initial data, (1.3) possesses a global weak solution, which decays exponentially for $\alpha = 2$ and in a polynomial rate if $\alpha > 2$. We first state an existence result, which can be established by repeating the same procedure of [4]. See also [1] and [3] for more standard results concerning local existence.

Proposition *Suppose that* $p \geq 2$, *such that*

$$\begin{aligned} 2 &\leq p \leq 1 + \frac{\alpha}{2}\frac{n}{n-\alpha}, \quad n \geq \alpha \\ p &\geq 2, \quad n < \alpha \end{aligned} \tag{1.4}$$

and let $u_0 \in W_0^{1,\alpha}(\Omega)$ *be given. Then problem* (1.3) *has a unique solution*

$$\begin{aligned} u &\in C\left([0,\ T);\ W_0^{1,\alpha}(\Omega)\right) \\ u_t &\in L^2\left(\Omega\times(0,\ T)\right). \end{aligned} \tag{1.5}$$

for some T *small.*

2. MAIN RESULT

In order to state and prove our main result we remind that by the embedding theorem there exists a constant C_* depending on Ω, p and α only such that

$$||u||_p \le C_* ||\nabla u||_\alpha. \tag{2.1}$$

We also introduce the following

$$\begin{aligned} I(t) &= I(u(t)) = ||\nabla u(t)||_\alpha^\alpha - b||u(t)||_p^p \\ E(t) &= E(u(t)) = \frac{1}{\alpha}||\nabla u(t)||_\alpha^\alpha - \frac{b}{p}||u(t)||_p^p \\ W &= \{w \in W_0^{1,\alpha}(\Omega):\ I(w) > 0\} \cup \{0\}. \end{aligned} \tag{2.2}$$

Remark 2.1. By multiplying equation (1.3) by u_t, integrating over Ω, and using integration by parts, we get

$$E'(t) = -||u_t(t)||_2^2 \le 0, \qquad \forall t \in [0,T). \tag{2.3}$$

Lemma 2.1. *Suppose that* (1.4) *holds. If* $u_0 \in W$ *satisfies*

$$\beta = bC_*^p \left(\frac{\alpha p}{p-\alpha} E(u_0)\right)^{(p-\alpha)/\alpha} < 1 \tag{2.4}$$

then $u(t) \in W$*, for each* $t \in [0,T)$.

Proof. Since $u_0 \in W$ then $I(u_0) > 0$. This implies the existence of $T_m \le T$ such that $I(u(t)) \ge 0$ for all $t \in [0, T_m)$. This implies

$$\begin{aligned} E(t) &= \frac{1}{\alpha}||\nabla u(t)||_\alpha^\alpha - \frac{b}{p}||u(t)||_p^p \\ &= \frac{p-\alpha}{\alpha p}||\nabla u(t)||_\alpha^\alpha + \frac{1}{p} I(u(t)) \\ &\ge \frac{p-\alpha}{\alpha p}||\nabla u(t)||_\alpha^\alpha, \qquad \forall t \in [0, T_m); \end{aligned} \tag{2.5}$$

hence

$$||\nabla u(t)||_\alpha^\alpha \le \frac{\alpha p}{p-\alpha}E(t) \le \frac{\alpha p}{p-\alpha}E(u_0), \quad \forall t \in [0,T_m). \tag{2.6}$$

By exploiting (2.1) and (2.6), we easily arrive at

$$\begin{aligned} b||u(t)||_p^p &\le bC_*^p||\nabla u(t)||_\alpha^p = bC_*^p||\nabla u(t)||_\alpha^{p-\alpha}||\nabla u(t)||_\alpha^\alpha \\ &\le bC_*^p\left(\frac{\alpha p}{p-\alpha}E(u_0)\right)^{(p-\alpha)/\alpha}||\nabla u(t)||_\alpha^\alpha = \beta||\nabla u(t)||_\alpha^\alpha \\ &< ||\nabla u(t)||_\alpha^\alpha, \qquad \forall t \in [0,T_m); \end{aligned} \tag{2.7}$$

hence $||\nabla u(t)||_\alpha^\alpha - b||u(t)||_p^p > 0$, $\forall t \in [0,T_m)$. This shows that $u(t) \in W$, $\forall t \in [0,T_m)$. By repeating the procedure, T_m is extended to T.

Theorem 2.2. *Suppose that* (1.4) *holds. If* $u_0 \in W$ *satisfying* (2.4) *Then the solution is global*

Proof. It suffices to show that $||\nabla u(t)||_\alpha^\alpha$ is bounded independently of t. To achieve this we use (2.2) and (2.3)

$$\begin{aligned} E(u_0) &\ge E(t) = \frac{1}{\alpha}||\nabla u(t)||_\alpha^\alpha - \frac{b}{p}||u(t)||_p^p \\ &= \frac{p-\alpha}{\alpha p}||\nabla u(t)||_\alpha^\alpha + \frac{1}{p}I(u(t)) \ge \frac{p-\alpha}{\alpha p}||\nabla u(t)||_\alpha^\alpha \end{aligned} \tag{2.8}$$

since $I(u(t)) \ge 0$. Therefore

$$||\nabla u(t)||_\alpha^\alpha \le \frac{\alpha p}{p-\alpha}E(u_0) \tag{2.9}$$

Theorem 2.3. *Suppose that* (1.4) *holds. Then there exist positive constants* K *and* k *such that, for all* $t \ge 0$*, the global solution of* (1.3) *satisfies*

$$\begin{aligned} E(t) &\le Ke^{-kt}, && \alpha = 2 \\ E(t) &\le (kt+K)^{-2/(\alpha-2)}, && \alpha > 2. \end{aligned} \tag{2.10}$$

Proof. We define

$$H(t) := E(t) + \frac{1}{2}\int_\Omega u^2(t)dx, \tag{2.11}$$

hence we have $E(t) \leq H(t)$ and

$$\begin{aligned} H(t) &\leq E(t)+\frac{1}{2}C_*^2\|\nabla u(t)\|_\alpha^2 \\ &\leq E(t)+\frac{1}{2}C_*^2\left(\frac{\alpha p}{p-\alpha}E(t)\right)^{2/\alpha} \\ &\leq \left(E^{1-(2/\alpha)}(u_0)+\frac{1}{2}C_*^2\left(\frac{\alpha p}{p-\alpha}\right)^{2/\alpha}\right)E^{2/\alpha}(t) \\ &= cE^{2/\alpha}(t). \end{aligned} \tag{2.12}$$

We differentiate (2.11) and use equation (1.3) and (2.3) to obtain

$$H'(t) = -\int_\Omega |u_t(t)|^2 dx - \int_\Omega |\nabla u(t)|^\alpha dx + b\int_\Omega |u(t)|^p dx. \tag{2.13}$$

We then use (2.2) and (2.7) to get

$$\begin{aligned} b\int_\Omega |u(t)|^p dx &= \lambda b\int_\Omega |u(t)|^p dx + (1-\lambda)b\int_\Omega |u(t)|^p dx \\ &= \lambda\left(\frac{p}{\alpha}\int_\Omega |\nabla u(t)|^\alpha dx - pE(t)\right) \\ &\quad +(1-\lambda)\beta\int_\Omega |\nabla u(t)|^\alpha dx, \quad 0<\lambda<1. \end{aligned} \tag{2.14}$$

Therefore a combination of (2.13) and (2.14) gives

$$\begin{aligned} H'(t) &\leq -\int_\Omega u_t^2(t)dx - \lambda pE(t) \\ &\quad +\varepsilon[\lambda(\frac{p}{\alpha}-1)-\eta(1-\lambda)]\int_\Omega |\nabla u(t)|^\alpha dx, \end{aligned} \tag{2.15}$$

where $\eta = 1-\beta$. By using (2.8) and choosing λ close to 1 so that

$$\lambda(\frac{p}{\alpha}-1)-\eta(1-\lambda) > 0,$$

we arrive at

$$\begin{aligned} H'(t) &\leq -\int_\Omega u_t^2(t)dx - \lambda pE(t) \\ &\quad +[\lambda(\frac{p}{\alpha}-1)-\eta(1-\lambda)]\frac{\alpha p}{p-\alpha}E(t) \\ &\leq -\int_\Omega u_t^2(t)dx - \eta(1-\lambda)\frac{\alpha p}{p-\alpha}E(t). \end{aligned} \tag{2.16}$$

We then recall (2.12) to obtain, from (2.16),

$$H'(t) \leq -\eta(1-\lambda)\frac{\alpha p}{p-\alpha}c^{-\alpha/2}H^{\alpha/2}(t). \tag{2.17}$$

We distinguish two cases.
i) $\alpha = 2$, then a simple integration of (2.17) leads to

$$E(t) \leq H(t) \leq H(0)e^{-kt}, \quad \forall t \geq 0, \tag{2.18}$$

where $k = \frac{\eta}{c}(1-\lambda)\frac{\alpha p}{p-\alpha}$.

ii) $\alpha > 2$, again a simple integration of (2.17) yields

$$E(t) \leq H(t) \leq \left(kt + H^{(2-\alpha)/2}(0)\right)^{-2/(\alpha-2)}, \tag{2.19}$$

where

$$k = (\frac{\alpha}{2} - 1)\eta(1-\lambda)\frac{\alpha p}{p-\alpha}c^{-\alpha/2}.$$

This completes the proof.

Remark 2.2. By using (2.5), (2.8), (2.18), and (2.19), we easily obtain, for all $t \geq 0$,

$$\begin{aligned} ||\nabla u(t)||_\alpha &\leq Ce^{-kt/2}, \qquad \alpha = 2 \\ ||\nabla u(t)||_\alpha &\leq C\,(t+1)^{-2/(\alpha-2)\alpha}, \qquad \alpha > 2 \end{aligned} \tag{2.20}$$

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REFERENCES

1. Chipot M., Element of Nonlinear Analysis, *Birkhauser Advanced Texts, Basler Lehrbucher*, first Edition, 2000

2. Englern H., B. Kawohl, and S. Luckhaus, Gradient estimates for solutions of parabolic equations and systems, *J. Math. Anal. Appl.* **147** (1990), 309 - 329

3. Friedman A., Partial differential equations of parabolic type, Prentice Hall Englewood NJ 1964.

4. Georgiev, V. and G. Todorova, Existence of solutions of the wave equation with nonlinear damping and source terms, *J. Diff. Eqns.* **109** (1994), 295 - 308.

5. Kalantarov V. K. and Ladyzhenskaya O. A., Formation of collapses in quasilinear equations of parabolic and hyperbolic types. (Russian) Boundary value problems of mathematical physics and related questions in the theory of functions, 10. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **69** (1977), 77–102.

6. Levine H., Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$, *Archive Rat.Mech. Anal.* **51** (1973), 371 - 386.

7. Levine H., S. Park, and J. Serrin, Global existence and nonexistence theorems for quasilinear evolution equations of formally parabolic type *J. Diff. Eqns* **142** (1998), 212 - 229.

8. Nakao M. and Y. Ohara, Gradient estimates for a quasilinear parabolic equation of the mean curvature type, *J. Math. Soc. Japan* **48 # 3** (1996), 455 - 466.

9. Nakao M. and C. Chen, Global existence and gradient estimates for the quasilinear parabolic equations of m-Laplacian type with a nonlinear convection term, *J. Diff. Eqns* **162** (2000), 224 - 250.

10. Pucci P. and J. Serrin, asymptotic stability for nonlinear parabolic systems, *Energy methods in continuum mechanics*, (Oviedo, 1994), 66–74, Kluwer Acad. Publ., Dordrecht, 1996.

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