Boundary stabilization in thermoelasticity of type III

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1 Introduction

Classical thermoelasticity

$$u_{tt} - \mu\triangle u - (\mu + \lambda)\nabla(divu) + \beta\nabla\theta = 0$$
$$\theta_t - \Delta\theta + \gamma divu_t = 0$$

The heat flux is given by Fourier's law.

$$q = -\kappa\nabla\theta$$

- This theory predicts an infinite speed of heat propagation
- It excluded the thermal pulse transmission
- Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox
- Disturbances, which are almost entirely thermal, propagate in a finite speed.

Thermoelasticity with second sound

Fourier' s law replaced by so called Cattaneo' s law

$$\tau(\theta)q_t + q = -\kappa(\theta)\nabla\theta$$

Existence, stability, and blow up results: Tarabek, Saouli, Racke, Messaoudi

Thermoelasticity Type III

Green and Naghdi introduced three types of thermoelastic theories based on an entropy equality instead of the usual entropy inequality. In each of these theories, the heat flux is given by a different constitutive assumption. As a results, three theories are obtained and were called thermoelasticity type I, type II, and type III respectively. This theory is developed in a rational way in order to obtain a fully consistent theory, which will incorporate thermal pulse transmission in a very logical manner and elevate the unphysical infinite speed of heat propagation induced by the classical theory of heat conduction. When the theory of type I is linearized the parabolic equation of the heat conduction arises. Whereas the theory of type II does not admit dissipation of energy and it is known as thermoelasticity without dissipation. In fact, it is a limiting case of thermoelasticity type III.

Contributions:

1) Survey paper of Chandrasekharaiah

- focussed attention on the work done during the last 10 or 12 years.

- reviewed the theory of thermoelasticity with thermal relaxation and the temperature-rate dependent thermoelasticity.

- described the thermoelasticity without dissipation
- made a brief discussion to the new theories, including what is called dual-phase-lag effects.

2) Zhang and Zuazua analyzed the long time behavior of the solution of the system

$$u_{tt} - \mu \triangle u - (\mu + \lambda)\nabla(divu) + \beta\nabla\theta = 0$$
$$\theta_{tt} - \Delta\theta + divu_{tt} - \Delta\theta_t = 0$$
$$u(.,0) = u_0,\, u_t(.,0) = u_1,\, \theta(.,0) = \theta_0,\, \theta_t(.,0) = \theta_1$$
$$u = \theta = 0$$

Conclusions:

- For most domains, the energy of the system does not decay uniformly.

- Under suitable conditions on the domain, which might be described in therms of Geometric Optics, the energy of the system decays exponentially.

- For most domains in two space dimension, the energy of smooth solutions decays in a polynomial rate.

3) Quintanilla and Racke used the spectral analysis method and the energy method to obtain the exponential stability in one dimension for different boundary conditions

- proved a decay of energy result for the radially symmetric situations. in multi- dimensional case

4) Quintanilla proved that solutions of thermoelasticity of type III converge to solutions of the classical thermoelasticity as well as to the solution of thermoelasticity without energy dissipation

5) Quintanilla established a structural stability result on the coupling coefficients and continuous dependence on the external data in thermoelasticity type III.

2 Our problem

Consider the problem

$$\begin{aligned} u_{tt} - \mu\triangle u - (\mu+\lambda)\nabla(divu) + \beta\nabla\theta = 0 \\ c\theta_{tt} - \kappa\Delta\theta + \beta divu_{tt} - \delta\Delta\theta_t = 0 \qquad (.1) \\ u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad \theta_t(.,0) = \theta_1 \\ u = 0, \quad x \in \Gamma_0, \quad t \geq 0 \\ \mu\frac{\partial u}{\partial\nu} + (\mu+\lambda)(divu)\nu = -au_t, \quad x \in \Gamma_1, \quad t \geq 0 \\ \theta = 0, \quad x \in \partial\Omega\,, t \geq 0, \end{aligned}$$

for $c, \delta, \kappa, \beta, \lambda, \mu$ positive constants, Ω a bounded domain of $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$. such that $\{ \Gamma_0 \cup \Gamma_1 \}$ is a partition of $\partial\Omega$, ν is the outward normal to $\partial\Omega$, $u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, and $\theta = \theta(x,t)$ is the difference temperature.

Hypotheses

There exists x_0 in $\mathbb{R}^n$, for which $m(x) = x - x_0$ satisfies

$$\inf_{x\in\Gamma_1} m(x).\nu > 0 \text{ and } m(x).\nu \leq 0, \quad \forall x \in \Gamma_0, \qquad (.2)$$

$$m(x).\nu > \eta > 0 \text{ on } \Gamma_1. \qquad (.3)$$

Let $v = u_t$ then problem (1) becomes

$$v_{tt} - \mu\triangle v - (\mu+\lambda)\nabla(divv) + \beta\nabla\theta_t = 0 \qquad (.4)$$

$$c\theta_{tt} - k\Delta\theta + \beta divv_t - \delta\Delta\theta_t = 0 \qquad (.5)$$

$$v(.,0) = u_1, \quad v_t(.,0) = (\lambda+\mu)\nabla\left(divu_0\right)+\mu\triangle u_0-\beta\nabla\theta_0$$

$$\theta(.,0) = \theta_0, \quad \theta_t(.,0) = \theta_1, \quad x \in \Omega$$

$$v\left(x,t\right) = 0, \quad x \in \Gamma_0, \quad t \geq 0$$

$$\mu\frac{\partial v}{\partial\nu} + (\mu+\lambda)(divv)\nu = -av_t, \quad x \in \Gamma_1, \quad t \geq 0$$

$$\theta = 0, \quad x \in \partial\Omega\,, t \geq 0.$$

The associated energy is defined by

$$E\left(t\right) = \frac{1}{2}\left[\int_\Omega \left|v_t\right|^2 dx + c\int_\Omega \theta_t^2 dx + \mu\int_\Omega \left|\nabla v\right|^2 dx\right]$$

$$+\frac{1}{2}\left[(\mu+\lambda)\int_\Omega (divv)^2 dx + k\int_\Omega \left|\nabla\theta\right|^2 dx\right] \qquad (.6)$$

Lemma 2.1: *The energy (6) of the solution is nonincreasing and satisfies:*

$$\frac{d}{dt}E\left(t\right) = -a\int_{\Gamma_1}\left|v_t\right|^2 d\sigma - \delta\int_\Omega \left|\nabla\theta_t\right|^2 dx$$

$$\leq 0, \ \forall t \geq 0. \qquad (.7)$$

Proof. A multiplication of equations (4) by v_t and (5) by θ_t respectively, and integration over Ω, using integration by parts and the boundary conditions, we obtain the assertion of the lemma.

Theorem 2.2 *Assume that* (2) *and* (3) *hold . Then there exist positive constants* C *and* ζ *independent of* t *such that*

$$E(t) \leq Ce^{-\zeta t}. \tag{.8}$$

Proof. We define

$$H\left(t\right) = E\left(t\right) + \varepsilon F\left(t\right) \tag{.9}$$

for $\varepsilon > 0$ to be chosen later and

$$\begin{aligned} F\left(t\right) &= \int_{\Omega} \left[M\left(x\right).v_t + (n-1)v.v_t\right] dx \\ &\quad + c\int_{\Omega} \theta\theta_t dx + \beta\int_{\Omega} \theta divv dx, \end{aligned} \tag{.10}$$

where

$$\begin{aligned} M &= \left(M_{1,}M_2, ..., M_n\right)^T \\ M_i &= 2m\left(x\right).\nabla v_i, \quad i = 1, ..n. \end{aligned}$$

Remark 2.1. It is easy to show that for ε small enough, $H\left(t\right)$ and $E\left(t\right)$ are equivalent.

By differentiating (10) and estimating each term, we arrive at

$$F^{'}\left(t\right) \leq -\left[(\lambda+\mu) - \frac{\beta(3n-2)\gamma_3}{2} - \beta\gamma_4\right]\int_{\Omega}\left(divv\right)^2 dx$$

$$
\begin{aligned}
&-\left[\mu - C_1^2\frac{a\,(n-1)\,\gamma_1}{2}\right]\int_\Omega |\nabla v|^2\,dx - \int_\Omega |v_t|^2\,dx \\
&+\left[\frac{\beta C_1\,(3n-2)}{2\gamma_3} + \frac{\beta R^2}{\gamma_4} + \frac{\delta}{2\gamma_5}\right]\int_\Omega |\nabla\theta_t|^2\,dx \\
&+\left[R + \frac{a}{\gamma_2} + \frac{a\,(n-1)}{2\gamma_1}\right]\int_{\Gamma_1} |v_t|^2\,d\sigma \\
&+\left(aR^2\gamma_2 - \mu\eta\right)\int_{\Gamma_1} |\nabla v|^2\,d\sigma \qquad (.11)\\
&+c\int_\Omega |\theta_t|^2\,dx + \left(\frac{\delta\gamma_5}{2} - k\right)\int_\Omega |\nabla\theta|^2\,dx.
\end{aligned}
$$

where C_1 is the Poincaré constant, $\gamma_1 - \gamma_5$ are constants coming from Young's inequality, and

$$
R = \max_{x\in\Omega} m(x)
$$

At this point we choose $\gamma_1 - \gamma_5$ small so that (11) becomes

$$
\begin{aligned}
F^{'}(t) \;\leq\; & -k_1\int_\Omega (divv)^2\,dx - k_2\int_\Omega |\nabla v|^2\,dx \\
& -\int_\Omega |v_t|^2\,dx + k_3\int_\Omega |\nabla\theta_t|^2\,dx \qquad (.12)\\
& +k_4\int_{\Gamma_1} |v_t|^2\,d\sigma + c\int_\Omega |\theta_t|^2\,dx - k_5\int_\Omega |\nabla\theta|^2\,dx
\end{aligned}
$$

where k_1, k_2, k_3, k_4 and $k_5$are strictly positive constants.

Combining (7) , (9) and (12) we conclude

$$\begin{aligned} H^{'}(t) \leq & -\varepsilon k_1 \int_\Omega (divv)^2 \, dx - \varepsilon k_2 \int_\Omega |\nabla v|^2 \, dx \\ & -\varepsilon \int_\Omega |v_t|^2 \, dx - (\delta - \varepsilon k_3) \int_\Omega |\nabla \theta_t|^2 \, dx \\ & - (a - k_4 \varepsilon) \int_{\Gamma_1} |v_t|^2 \, d\sigma + \varepsilon c \int_\Omega |\theta_t|^2 \, dx \\ & -\varepsilon k_5 \int_\Omega |\nabla \theta|^2 \, dx. \end{aligned} \tag{.13}$$

We choose ε small so that (13) becomes

$$\begin{aligned} H^{'}(t) \leq & -\eta_1 \int_\Omega (divv)^2 \, dx - \eta_2 \int_\Omega |\nabla v|^2 \, dx \\ & -\eta_3 \int_\Omega |\nabla \theta_t|^2 \, dx - \eta_4 \int_\Omega |\nabla \theta|^2 \, dx \\ -\varepsilon \int_\Omega |v_t|^2 \, dx \leq & -\mu E(t) \leq -\zeta H(t) \end{aligned}$$

for $\eta_1 - \eta_4$ and μ, ζ are strictly positive constants. A simple integration then leads to

$$H(t) \leq H(0) \, e^{-\zeta t}.$$

By using the fact that E and H are equivalent, the assertion of the theorem is established.