Lectures on some variational problems

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1 Premilinaries

In this section, we ptresent some classical result:s which serve us later on.

Theorem (Fixed point) Let (X, d) be a complete metric space and $S : X \longrightarrow X$ be an application satisfying

$d(Sv_1, Sv_2) \leq kd(v_1, v_2), \quad \forall v_1, v_2 \, \varepsilon \, X, \qquad 0 < k < 1.$

Then S possesses a unique fixed point u such that

$$Su = u.$$

Theorem (Riesz Representation) Let H be a Hilbert space and $H^{'}$ be its dual. Then for any $\phi \, \varepsilon \, H^{'}$ there exists a unique f in H which satisfies

$$\langle \phi, v \rangle = (f, v), \quad \forall v \, \varepsilon \, H.$$

2 Variational Problems

2.1 Hilbet Space Case

Definition 2.1. A bilinear form $a : H \times H \rightarrow \mathbb{R}$, on a Hilbert space H, is said to be continuous (or bounded) if there exists a constant $C > 0$, for which we have

$$|a(u, v)| \leq C||u||||v||, \qquad \forall u, v \varepsilon H$$

Definition 2.2. A bilinear form $a : H \times H \rightarrow \mathbb{R}$, on a Hilbert space H, is said to be coercive (elliptic) if there exists a constant $\alpha > 0$, for which we have

$$|a(u, u)| \geq \alpha||u||^2, \qquad \forall u \varepsilon H$$

Definition 2.3. A bilinear form $a : H \times H \rightarrow \mathbb{R}$, on a Hilbert space H, is said to be symmetric if

$$a(u, v) = a(v, u)$$

Remark 2.1. We will cross another definition for coercivity on Banach spaces.

Theorem 2.1. [2], [3]: (Stampacchia): Let K be a convex closed and nonempty subset of a Hilbert space H. Let $a : K \times K \rightarrow \mathbb{R}$ be a continous and coercive bilinear form. Then for any φ in H', there exists a unique $u \, \varepsilon \, K$ such that

$$a(u, v - u) \geq \langle \varphi, v - u \rangle, \quad \forall v \, \varepsilon \, K. \tag{.1}$$

Moreover, if a is symmetric then u is characterized by

$$u \, \varepsilon \, K$$

$$\frac{1}{2}\,a(u,u) - \langle \varphi, u\rangle = \min_{v\varepsilon\, K}\left\{\frac{1}{2}a(v,v) - \langle \varphi, v\rangle\right\}.$$

Proof. The Riesz representation theorem yields the existence of a unique element $f\,\varepsilon\, H$ such that

$$\langle \varphi, v\rangle = (f, v), \quad \forall v\,\varepsilon\, H.$$

In the other hand, for a fixed $u\,\epsilon\, H$, the map $v \longrightarrow a(u,v)$ is a continuous linear form over H. So, by the Riesz representation theorem there exists a unique element denoted by $Au\,\varepsilon\, H$ such that

$$a(u,v) = (Au, v), \quad \forall v\,\varepsilon\, H.$$

It is easy to verify that $A : H \longrightarrow H$ is linear and

$$||Au|| \leq C||u|| \quad \text{and } (Au,u) \geq \alpha||u||^2, \quad \forall u\,\varepsilon\, H.$$

Now, we would like to show that there exists $u\;\varepsilon\, K$ such that

$$(Au, v-u) \geq (f, v-u), \quad \forall v\,\varepsilon\, K. \qquad (.2)$$

Let $\delta > 0$ be a constant, to be determined later. The inequality (2) is equivalent to

$$(\delta f - \delta Au + u - u, v-u) \leq 0, \quad \forall v\,\varepsilon\, K. \qquad (.3)$$

This implies that

$$u = P_K(\delta f - \delta Au + u).$$

Now, we define the map $S : K \longrightarrow K$ by

$$Sv = P_K(\delta f - \delta Av + v).$$

From the properties of the projection operator we have

$$||Sv_1 - Sv_2|| \leq ||(v_1 - v_2) - \delta(Av_1 - Av_2)||$$

So,

$$\begin{aligned} ||Sv_1 - Sv_2||^2 &\leq ||v_1 - v_2||^2 + \delta^2 ||Av_1 - Av_2||^2 \\ &\quad -2\delta(Av_1 - Av_2, v_1 - v_2) \\ &\leq ||v_1 - v_2||^2 [1 - 2\delta\alpha + \delta^2 C^2]. \end{aligned}$$

By choosing δ so that $0 < \delta < \frac{2\alpha}{C^2}$ we arrive at $k = 1 - 2\delta\alpha + \delta^2 C^2 < 1$. Hence, S has a unique fixed point $u \,\varepsilon K$. It is clear that u is the solution of (3), hence to (2) or (1). Next, we suppose that a is symmetric. So the bilinear form a defines a new inner product on H with a norm $[a(u,u)]^{\frac{1}{2}}$ equivalent to the norm $|| \quad ||$. H is then a Hilbert space for this new inner product. Applying the Riesz representation theorem we obtain $g \,\varepsilon\, H$ such that

$$\langle \varphi, v \rangle = a(g, v), \quad \forall v \varepsilon\, H. \tag{.4}$$

So (1) is equivalent to

$$a(g - u, v - u) \leq 0, \quad \forall v \,\varepsilon\, K. \tag{.5}$$

This shows that $g = P_K u$. This projection is with respect to the new inner product. Therefore u is the solution for the problem

$$\min_{v \,\varepsilon\, K} [a(g - v, g - v)]^{\frac{1}{2}}$$

or

$$\min_{v \varepsilon\, K} a(g - v, g - v)$$

that is

$$\min_{v \,\varepsilon\, K} a(v, v) - 2a(g, v)$$

or

$$\min_{v\varepsilon K} \frac{1}{2} a(v,v) - < \varphi, v > \qquad (.6)$$

Corollary 2.2. [2], [3]: (Lax-Milgram)

Let $a : H \times H \rightarrow \mathbb{R}$ be a continuous and coercive bilinear form. Then, for $\varphi \, \varepsilon \, H'$ there exists $u \, \varepsilon \, H$ unique such that

$$a(u,v) = \langle \varphi, v \rangle, \quad \forall v \, \varepsilon \, H. \qquad (.7)$$

Moreover, if a is symmetric then u satisfies

$$u \, \varepsilon \, H, \quad \frac{1}{2} a(u,u) - \langle \varphi, u \rangle = \min_{v \, \varepsilon \, H} \left\{ \frac{1}{2} a(v,v) - \langle \varphi, v \rangle \right\}.$$

Proof. From Theorem 2.1, there exists a unique u in H, for which

$$a(u, w-u) \geq \langle \varphi, w-u \rangle, \quad \forall w \, \varepsilon \, H.$$

Take $w = v + u$, $\forall v \varepsilon \, H$. So we have

$$a(u,v) \leq \langle \varphi, v \rangle, \quad \forall v \, \varepsilon \, H \qquad (.8)$$

Take $w = -v + u, \quad \forall v \varepsilon \, H$. Thus we obtain

$$a(u,-v) \leq \langle \varphi, -v \rangle,$$

that is

$$a(u,v) \geq \langle \varphi, v \rangle \qquad (.9)$$

A combination of (8) and (9) yields (7).

The other part of the corollary follows directly from the previous theorem

Application 2.1 Given $f \,\varepsilon\, H^{-1}(\Omega) =$ dual of $H_0^1(\Omega)$. We seek a function u satisfying

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) = f & \text{in} \quad \Omega \\ u(x) = 0 & \text{on} \quad \Gamma = \partial\Omega. \end{cases}$$

Let v be in $H_0^1(\Omega)$ and define the bilinear form

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\ \frac{\partial v}{\partial x_j}\ dx$$

and the linear form $Fv =< f, v >$

Theorem 2.3. (Existence and uniqueness)

Let $a_{ij} \,\varepsilon L^\infty(\Omega), \quad \forall i, j = 1, \ldots n$ such that, for some $c_0 > 0$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2,$$

for almost every $x \,\varepsilon\, \Omega$ and for all $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \,\varepsilon\, \mathbb{R}^n$. Then for any $f \varepsilon\, H^{-1}(\Omega)$, there exists a unique $u \,\varepsilon\, H_0^1(\Omega)$ satisfying

$$a(u,v) = \langle f, v\rangle, \quad \forall v \,\varepsilon\, H_0^1(\Omega)$$

Proof. For the bilinear form a, we have

$$\begin{aligned} |a(u,v)| \ &\leq\ M \int_\Omega \sum_{i,j=1}^{n} \left|\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right| \\ &\leq\ C||\nabla u||_{L^2}\, ||\nabla v||_{L^2} \leq C||u||_{H_0^1}\, ||v||_{H_0^1}. \end{aligned}$$

and

$$\begin{aligned} a(u,u) &= \int_\Omega \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} \\ &\geq \int_\Omega c_0|\nabla u|^2 = c_0||u||^2_{H_0^1}. \end{aligned}$$

Here $H_0^1(\Omega)$ is equipped with its equivalent norm.

$$||u||_{H_0^1(\Omega)} = ||\nabla u||_{L^2(\Omega)}.$$

Therefore $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is a continuous and coercive. We also have

$$|\langle f, v\rangle| \leq ||f||_{H^{-1}}\,||v||_{H_0^1(\Omega)}$$

So $F : H_0^1(\Omega) \to \mathbb{R}$ is a continuous linear form. Therefore, Lax-Milgram lemma shows that $\exists$ a unique $u \,\varepsilon\, H_0^1(\Omega)$ such that

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\,dx = \langle f, v\rangle, \quad \forall v \,\varepsilon\, H_0^1(\Omega).$$

Remark 2.2. Suppose that Ω is of class C^2 (bounded)and $a_{ij} \,\varepsilon\, C^1(\bar{\Omega})$. If $f \,\varepsilon\, L^2(\Omega)$ then $u \,\varepsilon\, H^2(\Omega) \cap H_0^1(\Omega)$.

Remark 2.3. [2]: If a_{ij} and f are regular enough then u becomes smooth enough, so we obtain

$$\int_\Omega -\sum_{i,j}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)v = \int_\Omega fv, \quad \forall v \varepsilon\, C_0^\infty(\Omega);$$

hence

$$-\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = f \ \text{ for a.e. } x \,\varepsilon\, \Omega.$$

i.e. u becomes a classical solution.

2.2 Banach Space Case

Definition 2.4. Let V be a Banach space and $A : V \to V'$ be an operator. We say that A is montone if

$$\langle A(u) - A(v), \quad u - v\rangle \geq 0 \quad \forall u, v \,\varepsilon\, V.$$

If, in addition, A satisfies

$$\langle A(u) - A(v), \quad u - v\rangle > 0 \quad \forall u \neq v \,\varepsilon\, V$$

it is called strictly monotone.

Definition 2.5. Let V be a Banach space and $A : V \to V'$ be an operator. We say that A is hemicontinuous if we have, $\forall u, v, w \,\varepsilon\, V$, the function $\lambda \to \langle A(u + \lambda v), w\rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$.

Definition 2.6 [4], [5]: Let V be a Banach space. An operator $A : V \to V'$ is said to be pseudo-monotone if

(i) A is bounded.

(ii) When $u_j \rightharpoonup u$ in V and $\limsup\langle A(u_j), u_j - u\rangle \leq 0$ then

$$\liminf\langle A(u_j), u_j - v\rangle \geq \langle A(u), u - v\rangle \quad \forall v \,\varepsilon\, V.$$

Definition 2.7. Let V be a Banach space. $A : V \to V'$ is bounded if for any bounded set $S \subset V$, $A(S)$ is bounded in V'.

Proposition 2.4 [4]: Suppose that A is bounded, hemi-continuous, and monotone then A is pseudo-monotone.

Proof. Suppose that $u_j \rightharpoonup u$ and

$$\limsup\langle A(u_j), u_j - u\rangle \geq 0.$$

By using he monotonicity of A, we have

$$\langle A(u_j), u_j - u\rangle \geq \langle A(u), u_j - u\rangle \to 0 \qquad \text{as } _j \to +\infty.$$

So

$$\liminf\langle A(u_j), u_j - u\rangle \geq 0.$$

Consequently

$$0 \leq \lim inf\langle A(u_i), u_j - u\rangle \leq \limsup\langle A(u_j), u_j - u\rangle \leq 0,$$

which gives

$$\lim_{j\to\infty}\langle A(u_j), u_j - u\rangle = 0. \qquad (.10)$$

Next, for $\theta\,\varepsilon\,]0,\ \ 1[$, let $w = (1-\theta)u + \theta v$. So, we have

$$\langle A(u_j) - A(w), \quad u_j - w\rangle \geq 0$$

That is

$$\begin{aligned}\theta\langle A(u_j), u - v\rangle \ \geq\ & -\langle A(u_j), u_j - u\rangle \\ & +\langle A(w), u_j - u\rangle - \theta\langle A(w), v - u\rangle\end{aligned}$$

Thus, by using (10), we arrive at

$$\theta\liminf\langle A(u_j), u - v\rangle \geq -\theta\langle A(w), v - u\rangle,$$

which implies

$$\liminf\langle A(u_j), u - v\rangle \geq \langle A(w), u - v\rangle$$

By noting that
$\langle A(u_j), u_j - v\rangle = \langle A(u_j), u_j - u\rangle + \langle A(u_j), u - v\rangle$
then

$$\begin{aligned} \liminf\langle A(u_j), u_j - v\rangle &\geq \liminf\langle A(u_j), u - v\rangle \\ &\geq \langle A(w), u - v\rangle \end{aligned}$$

As $\theta \longrightarrow 0$, $w \longrightarrow u$, using continuity. So we get

$$\liminf\langle A(u_j), u_j - v\rangle \geq \langle A(u), u - v\rangle, \quad \forall v \varepsilon V.$$

and since it is bounded then it is pseudo-monotone. This completes the proof.

Theorem 2.5. . Let V be a reflexive and separable Banach space. Suppose that $A : V \longrightarrow V'$ has the following properties:

(i) A is monotone, bounded, and hemicontinuous.

(ii) $\frac{<A(v),v>}{||v||} \longrightarrow +\infty$ as $||v|| \longrightarrow +\infty$.

Then, for each $f \,\varepsilon V'$, $\exists u \,\varepsilon\, V$ such that $A(u) = f$. Moreover, if

$$\langle A(u) - A(v), u - v\rangle > 0, \quad \forall u, v \,\varepsilon\, V, \qquad u \neq v,$$

then u is anique.

Remark 2.4. Property (ii) is sometimes called coercivity.

Remark 2.5. For the proof, we refer to [4], p 171-173.

Application 2.2. Let Ω be a bounded domain of $\mathbb{R}^n$ with a smooth boundary we would like to solve the problem

$$A(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f, \qquad \text{in } \Omega$$
$$u = 0, \qquad \text{on } \partial\Omega.$$

for $1 < p < +\infty,\ \ f \,\varepsilon\, W^{-1,p'}(\Omega) =$ dual of $W_0^{1,p}(\Omega)$.
Solution Let us verify the conditions of Theorem 2.5 for $V = W_0^{1,p}(\Omega)$ equipped with the norm

$$||v|| = ||\nabla v||_{L^p}.$$

$$\begin{aligned} |\langle A(u), v\rangle| &= \left|\int_\Omega \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right| \\ &\le \sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-1}\left|\frac{\partial v}{\partial x_i}\right| \\ &\le C||\nabla u||_{L^p}^{p-1}||v||_{L^p} \end{aligned}$$

Therefore $||A(u)||_{V'} \le C||u||_V^{p-1}$; i.e A is bounded.
Note for $p = 2$, we have

$$A(u)||_{V'} \le C||u||_V.$$

<u>Hemicontinouty</u>

Let u, v, w be in $W_0^{1,p}(\Omega)$, We verify that :

$$\begin{aligned} g(t) &= \langle A(u+tv), w\rangle \\ &= \int_\Omega \sum_{i=1}^{n}\left|\frac{\partial(u+tv)}{\partial x_i}\right|^{p-2}\frac{\partial(u+tv)}{\partial x_i}\frac{\partial w}{\partial x_i} \end{aligned}$$

is continous from $\mathbb{R}$ to $\mathbb{R}$.

Let $t_n \to t$. So

$$\sum_{i=1}^{n} \left| \frac{\partial(u+t_nv)}{\partial x_i} \right|^{p-2} \frac{\partial(u+t_nv)}{\partial x_i} \frac{\partial w}{\partial x_i}$$
$$\to \sum_{i=1}^{n} \left| \frac{\partial(u+tv)}{\partial x_i} \right|^{p-2} \frac{\partial(u+tv)}{\partial x_i} \frac{\partial w}{\partial x_i}$$

for almost every x in Ω. Also,

$$\sum_{i=1}^{n} \left| \frac{\partial(u+t_nv)}{\partial x_i} \right|^{p-1} |\frac{\partial w}{\partial x_i}|$$
$$\leq 2^{p-1} \sum_{i=1}^{n} \left[\left| \frac{\partial u}{\partial x_i} \right|^{p-1} |\frac{\partial w}{\partial x_i}| + M \left| \frac{\partial v}{\partial x_i} \right|^{p-1} |\frac{\partial w}{\partial x_i}| \right]$$

and

$$\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p-1} |\frac{\partial w}{\partial x_i}| dx \leq ||\frac{\partial u}{\partial x_i}||_p^{p-1} ||\frac{\partial w}{\partial x_i}||_p$$
$$\int_\Omega \left| \frac{\partial v}{\partial x_i} \right|^{p-1} |\frac{\partial w}{\partial x_i}| dx \leq ||\frac{\partial v}{\partial x_i}||_p^{p-1} ||\frac{\partial w}{\partial x_i}||_p$$

We thus conclude, by the DCT, that

$$g(t_n) = \sum_{i=1}^{n} \left| \frac{\partial(u+t_nv)}{\partial x_i} \right|^{p-2} \frac{\partial(u+t_nv)}{\partial x_i} \frac{\partial w}{\partial x_i} \to g(t)$$

Monotonicity:

$$\langle A(u)-A(v), u-v\rangle = \int_\Omega \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right)$$
$$-\int_\Omega \sum_{i=1}^{n} \left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i}\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right)$$
By using this simple inequality
$$(|a|^{P-2}a - |b|^{P-2}b)(a-b) \geq 0,$$
we easily arrive at $\langle A(u) - A(v), u - v\rangle \geq 0$. Thus, A is monotone.

<u>Coercivity:</u>
$$\langle A(v), v\rangle = \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^{p} = ||\nabla v||_{L^p}^p = ||v||_V^p$$
Which implies that
$$\frac{< A(v), v >}{||v||_V} = ||v||_V^{p-1} \to \infty \text{ since } p > 1.$$
Therefore for any $f \,\varepsilon\, W^{-1,p}(\Omega)$, $\exists u \,\varepsilon\, W_0^{1,p}(\Omega)$ such that
$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} = \langle f, v\rangle, \quad \forall v \,\varepsilon\, W_0^{1,p}(\Omega).$$
<u>Uniqueness:</u> This follows from the fact that
$$\langle A(u) - A(v), u - v\rangle > 0, \quad \forall u, v, \,\varepsilon\, V, \quad u \neq v.$$

Variational Inequalities

Theorem 2.6 (bounded case) Suppose that K is a convex, closed, and bounded nonempty set of a Banach space V. Let A be psendo-monotone operator from K to V'. Then for each $f \,\varepsilon\, V'$ there exists $u \,\varepsilon\, K$ such that

$$\langle A(u), v-u\rangle \geq \langle f, v-u\rangle, \forall v \,\varepsilon\, V$$

Theorem 2.7 (Unbounded case): Suppose that K is a convex, closed, and unbounded nonempty set of a Banach space V. Let $A : K \to V'$ be a pseudo-monotone operator satisfying, for some $v_0 \,\varepsilon\, K$, the following coercivitgy property

$$\frac{< A(v), v - v_0 >}{||v||} \to \infty \text{ as } ||v|| \to +\infty, \quad v \,\varepsilon\, K.$$

Then for any $f \,\varepsilon\, V'$, there exists $u \,\varepsilon\, K$ such that

$$\langle A(u), v-u\rangle \geq \langle f, v-u\rangle, \quad \forall v \,\varepsilon\, K.$$

Remark 2.6. For the proofs, we refer to [4], p 245-248.

Corollary 2.8. If K is a subspace of V then we have, under the above conditions,

$$\langle A(u), v\rangle = \langle f, v\rangle, \quad \forall v \,\varepsilon\, K.$$

Application 2.3 Let $V = H_0^1(\Omega)$,

$$K = \{\phi \,\varepsilon V \ / \ \phi(x) \geq 0, \text{ for } a.e. \ x \,\varepsilon\, \Omega\}$$

K is convex, closed, and unbounded. Let $A : K \to V'$ be defined as $A(u) = -\Delta u$. We would like to solve the following variational inequality: Find u in V which satisfies

$$\langle A(u), v-u\rangle \geq \langle f, v-u\rangle, \quad \forall v\,\varepsilon\, K.$$

Solution Let us verify the conditions of Theorem 2.7.

Boundedness:

$$\begin{aligned}\langle A(u), v\rangle &= \int_\Omega \nabla u\cdot\nabla v \\ &\leq ||\nabla u||_{L^2}\nabla v||_{L^2} = ||u||_V||u||_V\end{aligned}$$

$||A||_{V'} \leq 1$ (in fact it is equal to 1).

Hemicontinuity For $t\,\varepsilon\,$ ℝ, it clear that

$$\begin{aligned}g(t) &= \int_\Omega \nabla(u+tv)\cdot\nabla w \\ &= \int_\Omega \nabla u\cdot\nabla w + t\int_\Omega \nabla v\cdot\nabla w\end{aligned}$$

is continuous

Monotonicity: It is simple to see that

$$\langle A(u)-A(v), u-v\rangle = \int_\Omega |\nabla(u-v)|^2 \geq 0$$

In fact, we have

$$\langle A(u)-A(v), u-v\rangle > 0, \quad \forall u\neq v\,\varepsilon\, V.$$

We conclude that $A : V \to V'$ is ”strictly” pseudo-monotone.

Coercivity: Let $v_0 = 0\,\varepsilon\, K$, so

$$\langle A(v), v\rangle = ||v||_V^2$$

implies

$$\frac{< A(v), v >}{||v||} = ||v|| \to +\infty \text{ as } ||v|| \to +\infty,\, v\,\varepsilon\,K.$$

Therefore, there exists a unique $u\,\varepsilon\,K$ such that

$$\begin{aligned}\langle -\Delta u, v-u\rangle &= \int_\Omega \nabla u \cdot \nabla(v-u) \\ &\geq \langle f, v-u\rangle, \quad \forall v\,\varepsilon\,V.\end{aligned}$$

Application 2.4. In Application 2.3 let

$$K = \left\{\phi\,\varepsilon H_0^1(\Omega) \;/\; |\nabla\,\phi| \leq 1, \quad for \text{ a.e. } x\,\varepsilon\,\Omega\right\}$$

for Ω bounded and smooth. Thus, K is bounded in $V = H_0^1(\Omega)$ equipped with $||v||_V = ||\nabla v||_{L^2}$. Also K is nonempty $(0\,\epsilon\,K)$, and closed.

By applying Theorem 2.6, we easily conclude that for any $f\,\varepsilon\,L^2(\Omega)$ there exists a unique $u\,\varepsilon\,K$ for which, we have

$$\int_\Omega \nabla u \cdot \nabla(v-u) \geq \int_\Omega f(v-u), \quad \forall v\,\varepsilon\,K.$$

Remark 2.7. For some nonlinear problems, we refer the reader to [5].

2.3 Regularity

Consider the problem model

$$a(u,v) = \sum_{i,j} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}$$

in Ω, a bounded set of C^∞ regular boundary.

It is well known if

$$\sum_{i,j} a_{i_j} \xi_i \xi_j \geq c_0 |\xi|^2, \quad \forall \xi \, \epsilon \, \mathbb{R}^n$$

and f in $H^k(\Omega), k \geq 0$, then there exists u in $H_0^1(\Omega) \cap H^{k+2}(\Omega)$ unique satisfying

$$a(u,v) = \int_\Omega fv, \quad \forall v \, \varepsilon \, C_0^\infty(\Omega).$$

This is called the elliptic regularity (see [1], [2]). Unfortunately, it is not the case for the variational inequality.

Counter-example 2.5. [4]: Let $\Omega = (0,1)$, $f(x) = 4$, which is in $H^k(\Omega), \forall k \geq 0$. Let $a(u,v) = \int_0^1 u'v'dx$ and

$$K = \{v \, \varepsilon \, H_0'(\Omega) \; / \; |v'(x)| \leq 1, \text{ for a.e. } x \, \varepsilon \, (0,1)\}.$$

This is a closed and bounded convex set $(|v(x)| \leq 1, \; \forall x \, \varepsilon \, \Omega)$. The variational inequality

$$a(u, v-u) \geq \int_\Omega f(v-u), \; \forall v \, \varepsilon \, K \qquad (.11)$$

has a uniquesolution. In fact, it is easy to check that

$$u(x) = \begin{cases} x, & 0 < x \leq \frac{1}{4} \\ -2x^2 + 2x - \frac{1}{8}, & \frac{1}{4} < x \leq \frac{3}{4} \\ 1 - x & \frac{3}{4} < x < 1 \end{cases}$$

is a member of K and satisfies the inquality (11) since

$$a(u, v - u) = 4 \int_{\frac{1}{4}}^{\frac{3}{4}} v(x)dx - \frac{2}{3}, \quad \forall v \,\varepsilon\, H_0^1(\Omega)$$

and

$$\begin{aligned} \int_0^1 f(v-u) &= 4 \int_0^1 v(x)dx - 4 \int_0^1 u(x)dx \\ &= 4 \int_0^1 v(x)dx - 4 \left[\frac{1}{8} + \frac{1}{32} + \frac{2}{3} \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} & a(u, v-u) - \int_\Omega f(v-u) \\ &= 2 + \frac{5}{8} - 4 \left[\int_0^{\frac{1}{4}} v(x)dx + \int_{\frac{3}{4}}^1 v(x)dx \right] \\ &\geq 2 + \frac{5}{8} - 4 \left[\frac{1}{4} + \frac{3}{4} \right] = \frac{5}{8} > 0. \end{aligned}$$

However, $u \notin H^3(\Omega)$.

Remark 2.8. The question is, if $f \,\varepsilon\, X$, a subspace of V'. What would be the condition on X, K, and the operator A so that if u is the solution of the inequality then $A(u) \,\varepsilon\, X$?

Remark 2.9. Following the proofs in the regularity theory for elliptic equations [1], [2], one can easily see that all the classical results remain valid if K is a ball, for instance, of $H_0^1(\Omega)$, that is

$$K = \left\{ v \, \epsilon \, H_0^1(\Omega) \; / \; |v||_{H_0^1(\Omega)} \leq M \right\}.$$

3 References

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