On the decay of solutions in an abstract integro-differential equation

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1 Our problem:

$$u_{tt}(t)+Au(t)-\int_0^t g(t-s)Au(s)ds+f(u)=0, \qquad t>0$$
$$u(0)=u_0\in V, \qquad u_t(0)=u_1\in H, \qquad (1)$$

where $A : V \longrightarrow V'$ is a “differential” operator.

Hypotheses:

(H1) There exixts an operator $B : V \longrightarrow H$ such that

$$< Au, v >_{V'\times V} = < Bu, Bv >_{H\times H} \qquad (2)$$

(H2) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function satisfying

$$g(0)>0, \qquad 1-\int_0^\infty g(s)ds = l > 0 \qquad (3)$$
$$g'(t)\le -\xi g^p(t),\ t\ge 0, \quad 1\le p<\frac{3}{2}. \qquad (4)$$

(H3) There exists a constant $C_p > 0$ such that

$$||v||^2 \leq C_p||Bv||^2, \qquad \forall v \in V, \tag{5}$$

where $||.||$ is the norm in H.
(H4) $f : V \rightarrow H$ such that

$$||f(v)|| \leq C_p||Bv||^\alpha, \qquad \forall v \in V, \quad \alpha \geq 1$$

(H5) There exists $F : V \rightarrow \mathbb{R}_+$ satisfying

$$\begin{aligned} \frac{d}{dt}F(u(t)) &= < f(u(t)), u_t > \\ F(u(t)) - &< f(u(t)), u > \leq 0 \end{aligned}$$

Definition: By a weak solution of (1), we mean a function

$$u \in C([0,T); V) \cap C^1([0,T); H)$$

satisfying, for almost every $t \geq 0$,

$$\begin{aligned} &\frac{d}{dt} < u_t(t), v > + < Bu(t), Bv > \\ &+ < f(u(t)), v > - \int_0^t g(t-s) < Bu(s), Bv > ds \end{aligned}$$

$$+ < f(u(t)), v >= 0, \quad \forall v \in V$$
$$u(0) = u_0 \in V, \qquad u_t(0) = u_1 \in H$$

The energy

$$\mathcal{E}(t) = \frac{1}{2}\left(1 - \int\limits_0^t g(s)ds\right)||Bu(t)||^2 \tag{6}$$
$$+\frac{1}{2}||u_t||^2 + \frac{1}{2}(g \circ Bu)(t) + F(u(t)),$$

where

$$(g \circ v)(t) = \int\limits_0^t g(t-\tau)||v(t) - v(\tau)||^2 d\tau \tag{7}$$

Remark. Condition $p < 3/2$ is made so that

$$\int_0^\infty g^{2-p}(s)ds < \infty.$$

2 Decay of solutions

Let

$$L(t) := \mathcal{E}(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t), \qquad (8)$$

$\varepsilon_1, \varepsilon_2 > 0$ and

$$\Psi(t) \ : =< u, u_t >_{H\times H} \qquad (9)$$

$$\chi(t) \ : = - < u_t, \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau >_{H\times H} .$$

Lemma 2.1 *For* $r > 1$ *and* $0 < \theta < 1$, *we have*

$$\int_0^t g(t-s)||w(s)||^2 ds \leq \left(\int_0^t g^{1-\theta}(t-s)||w(s)||^2 ds \right)^{1/r}$$
$$\times \left(\int_0^t g^{(r-1+\theta)/(r-1)}(t-s)||w(s)||^2 ds \right)^{r/(r-1)}$$

Proof. It suffice to note that

$$\int_0^t g(t-s)||w(s)||^2 ds =$$

$$\int_0^t g^{(1-\theta)/r}(t-s)||w(s)||^{2/r} g^{(r-1+\theta)/r}(t-s)||w(s)||^{2(r-1)/r} ds$$

and apply Holder's inequality.

Lemma 2.2. *Let* $v(t)$ *be such that* $Bv \in L^\infty((0,T);H)$ *and* g *be a continuous function on* $[0,T]$ *and suppose that* $0 < \theta < 1$ *and* $p > 1$. *Then, there exists a constant* $C > 0$ *such that*

$$\int_0^t g(t-s)||Bv(t) - Bv(s)||^2 ds \leq$$

$$C\left(\sup_{0<s<T} ||Bv(s)||^2 \int_0^t g^{1-\theta}(s)ds\right)^{\frac{p-1}{p-1+\theta}}.$$

$$\times \left(\int_0^t g^p(t-s)||Bv(t) - Bv(s)||^2 ds\right)^{\frac{\theta}{p-1+\theta}}$$

Proof. By using lemma 2.1 with

$$r = (p-1+\theta)/(p-1),$$

we obtain

$$\int_0^t g(t-s)||Bv(t) - Bv(s)||^2 ds \leq \tag{10}$$

$$\left(\int_0^t g^{1-\theta}(t-s)||Bv(t) - Bv(s)||^2 ds\right)^{\frac{p-1}{p-1+\theta}}$$

$$\times \left(\int_0^t g^p(t-s)||Bv(t)-Bv(s)||^2 ds \right)^{\frac{\theta}{p-1+\theta}}$$

It is easy to see that

$$\int_0^t g^{1-\theta}(t-s)||Bv(t)-Bv(s)||^2 ds \leq$$

$$C \sup_{0<s<T} ||Bv(s)||^2 \int_0^t g^{1-\theta}(s) ds \tag{11}$$

By combining (10) and (11), the proof of the lemma is complete.

Lemma 2.3. *Let* $v(t)$ *be such that* $Bv \in L^\infty((0,T);H)$ *and* g *be a continuous function on* $[0,T]$ *and suppose that* $p>1$*. Then, there exists a constant* $C>0$ *such that*

$$\int_0^t g(t-s)||Bv(t)-Bv(s)||^2 ds \leq \tag{12}$$

$$C\left(t||Bv(t)||^2+\int_0^t ||Bv(s)||^2 ds\right)^{(p-1)/p}$$

$$\times\left(\int_0^t g^p(t-s)||Bv(t)-Bv(s)||^2 ds\right)^{1/p}.$$

Proof. We use (10), for $\theta = 1$ to arrive at

$$\int_0^t g(t-s)||Bv(t)-Bv(s)||^2 ds \leq$$

$$\left(\int_0^t ||Bv(t)-Bv(s)||^2 ds\right)^{(p-1)/p}$$

$$\times\left(\int_0^t g^p(t-s)||Bv(t)-Bv(s)||^2 ds\right)^{1/p}.$$

It suffices to note that

$$\int_0^t ||Bv(t)-Bv(s)||^2 ds =$$

$$t||Bv(t)||^2 + \int\limits_0^t ||Bv(s)||^2 ds$$

to obtain (12). This completes the proof.

Lemma 2.4 *If* u *is the solution of* (1) *then the energy satisfies*

$$\begin{aligned} \mathcal{E}'(t) &= \frac{1}{2}(g' \circ Bu)(t) + g(t)||Bu(t)||^2 \\ &\leq \frac{1}{2}(g' \circ Bu)(t) \leq 0. \end{aligned} \tag{13}$$

Proof. By multiplying ”scalarly” equation (1) by u_t and using (2)-(4) and some manipulations, we obtain (13).

Lemma 2.5. *For* ε_1 *and* ε_2 *small enough, we have*

$$\alpha_1 L(t) \leq \mathcal{E}(t) \leq \alpha_2 L(t), \tag{14}$$

holds for two positive constants α_1 *and* α_2.

Proof. Straightforward computations, using (2), (5), lead to

$$L(t) \le \mathcal{E}(t) + (\varepsilon_1/2)\,||u_t||^2 + (\varepsilon_1/2)\,||u||^2$$

$$+ (\varepsilon_2/2)\,||u_t||^2 + (\varepsilon_2/2)\,||\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau||^2. \tag{15}$$

By using

$$\begin{aligned}
&||\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau|| \\
&\le \int_0^t g(t-\tau)||(u(t)-u(\tau))||d\tau \\
&\le \left(\int_0^t \left(\sqrt{g(t-\tau)}\right)^2 ||(u(t)-u(\tau))||^2 d\tau\right)^{1/2} \\
&\times \left(\int_0^t \left(\sqrt{g(t-\tau)}\right)^2 d\tau\right)^{1/2} \\
&= \left(\int_0^t g(t-\tau)||(u(t)-u(\tau))||^2 d\tau\right)^{1/2} \left(\int_0^t g(t-\tau)d\tau\right)^{1/2} \\
&\le ((1-l)(g \circ Bu)(t))^{1/2},
\end{aligned} \tag{16}$$

we arrive at

$$L(t) \le \mathcal{E}(t) + [(\varepsilon_1+\varepsilon_2)/2]\,||u_t||^2 + (\varepsilon_1/2)\,C_p||Bu||^2$$

$$+ (\varepsilon_2/2)\,C_p(1-l)(g \circ Bu)(t) \le \alpha_2 \mathcal{E}(t). \tag{17}$$

Similarly we have

$$L(t) \geq \left(\tfrac{1}{2} - [(\varepsilon_1 + \varepsilon_2)/2]\right) ||u_t||^2$$

$$+ \left(\tfrac{1}{2} - (\varepsilon_1/2)\, C_p\right) ||Bu(t)||^2 + F(u(t))$$

$$+[\tfrac{1}{2} - (\varepsilon_2/2)\, C_p(1-l)](g \circ Bu)(t) \geq \alpha_1 \mathcal{E}(t) \tag{18}$$

for ε_1 and ε_1 small enough.

Lemma 2.6 *Under the asumptions* (2)-(5), *the functional*

$$\Psi(t) := < u, u_t >_{H \times H}$$

satisfies, along the solution of (1)

$$\begin{aligned} \Psi'(t) \;&\leq\; ||u_t||^2 - \frac{l}{2}||Bu||^2 - F(u) \qquad (19) \\ &+\frac{1}{l}\left[\int_0^t g^{2-p}(\tau)d\tau\right](g^p \circ Bu)(t). \end{aligned}$$

Proof By using equation (1), we see:

$$\begin{aligned} \Psi'(t) \;&=\; < u, u_{tt} >_{H \times H} + ||u_t||^2 \\ &=\; ||u_t||^2 - ||Bu||^2 - < u, f(u) >_{H \times H} \qquad (20) \end{aligned}$$

$$+ < Bu, \int_0^t g(t-\tau)Bu(\tau)d\tau >$$

$$\leq \ ||u_t||^2 - ||Bu||^2 - F(u)$$

$$+ \ < \ Bu, \int_0^t g(t-\tau)Bu(\tau)d\tau >_{H\times H}$$

Estimate the forth term in the RS of (20) as follows:

$$< Bu, \int_0^t g(t-\tau)Bu(\tau)d\tau >_{H\times H}$$

$$\leq \frac{1}{2}||Bu||^2 + \frac{1}{2}|| \int_0^t g(t-\tau)Bu(\tau)d\tau||^2$$

$$\leq \ \frac{1}{2}||Bu||^2 + \tag{21}$$

$$\frac{1}{2}|| \int_0^t g(t-\tau)B(u(\tau) - u(t) + u(t))d\tau||^2$$

Use Cauchy-Schwarz, Young's inequalities, and

$$\int_0^t g(\tau)d\tau \leq \int_0^\infty g(\tau)d\tau = 1 - l,$$

to obtain, for any $\eta > 0$,

$$|| \int_0^t g(t-\tau)B(u(\tau) - u(t) + u(t))d\tau||^2$$

$$\leq || \int_0^t g(t-\tau)(Bu(\tau) - Bu(t))d\tau||^2$$

$$+|| \int_0^t g(t-\tau)Bu(t)d\tau||^2$$

$$+2 < \int_0^t g(t-\tau)B(u(\tau) - u(t))d\tau, \tag{22}$$

$$\int_0^t g(t-\tau)Bu(t)d\tau >$$

$$\leq (1+\eta)|| \int_0^t g(t-\tau)Bu(t)d\tau||^2$$

$$+(1+\frac{1}{\eta})|| \int_0^t g(t-\tau)(Bu(\tau) - Bu(t))d\tau||^2$$

Cauchy-Schwarz inequality again $\Longrightarrow$

$$
\begin{aligned}
&||\int_0^t g(t-\tau)(B(u(\tau)-u(t))d\tau||^2 \\
&= \left(\int_0^t g(t-\tau)||Bu(\tau)-Bu(t)||d\tau\right)^2 \\
&= \left(\int_0^t g^{1-p/2}g^{p/2}(t-\tau)||B(u(\tau)-u(t))||d\tau\right)^2 \\
&\leq \left(\int_0^t g^{2-p}(\tau)d\tau\right)\int_0^t g^p(t-\tau)||B(u(\tau)-u(t))||^2d\tau
\end{aligned}
\tag{23}
$$

Thus (22) becomes

$$
\begin{aligned}
&||\int_0^t g(t-\tau)B(u(\tau)-u(t)+u(t))d\tau||^2 \\
&\leq (1+\eta)\left(\int_0^t g(t-\tau)d\tau\right)^2||Bu(t)||^2 \\
&+(1+\tfrac{1}{\eta})\left(\int_0^t g^{2-p}(\tau)d\tau\right)(g^p\circ Bu)(t) \\
&\leq (1+\eta)(1-l)^2||Bu(t)||^2 \\
&+(1+\tfrac{1}{\eta})\left(\int_0^t g^{2-p}(\tau)d\tau\right)(g^p\circ Bu)(t).
\end{aligned}
\tag{24}
$$

Combining (20)-(24) $\Longrightarrow$

$$\Psi'(t) \;\le\; ||u_t||^2 + \frac{1}{2}\left[-1 + (1+\eta)(1-l)^2\right]||Bu||^2$$

$$+(1+\frac{1}{\eta})\left(\int_0^t g^{2-p}(\tau)d\tau\right)(g^p \circ Bu)(t) - F(u). \tag{25}$$

Choose $\eta = l/(1-l)$ so proof is completed.

Lemma 2.7 *Under the asumptions* (2)-(5), *the functional*

$$\chi(t) := - < u_t, \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau >$$

satisfies, along the solution of (1)

$$\chi'(t) \;\le\; \delta\{1 + 2(1-l)^2 + (\frac{\mathcal{E}(0)}{l})^{\alpha-1}\}||Bu||^2$$

$$+\{2\delta + \frac{1}{\delta}\}(\int_0^t g^{2-p}(\tau)d\tau)(g^p \circ \nabla u)(t)$$

$$+\frac{g(0)}{4\delta}C_p(-(g' \circ Bu)(t) \tag{26}$$

$$+\{\delta - \int_0^t g(s)ds\}||u_t||^2, \quad \forall \delta > 0,$$

Proof.

$$\chi'(t) := - < u_{tt}, \int\limits_0^t g(t-\tau)(u(t)-u(\tau))d\tau >$$
$$- < u_t, \int\limits_0^t g'(t-\tau)(u(t)-u(\tau))d\tau >$$
$$- \left(\int\limits_0^t g(\tau)d\tau\right) ||u_t(t)||^2$$
$$= - < Au(t), \int\limits_0^t g(t-\tau)(u(t)-u(\tau))d\tau >$$
$$+ < \int_0^t g(t-s)Au(s)ds, \int\limits_0^t g(t-\tau)(u(t)-u(\tau))d\tau >$$
$$< -f(u), \int\limits_0^t g(t-\tau)(u(t)-u(\tau))d\tau >$$
$$- < u_t, \int\limits_0^t g'(t-\tau)(u(t)-u(\tau))d\tau >$$
$$- \left(\int\limits_0^t g(\tau)d\tau\right) ||u_t(t)||^2$$

All the terms are treated in a similar way except

$$< -f(u), \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau >$$
$$\leq \delta ||f(u)||^2 + \frac{1}{4\delta}|| \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau||^2$$
$$\leq \delta C_p^2 ||Bu||^{2\alpha} + \frac{1}{4\delta}\left(\int_0^t g^{2-p}(\tau)d\tau\right)(g^p \circ Bu)(t)$$
$$\leq \delta C_p^2 (\frac{\mathcal{E}(0)}{l})^{\alpha-1} ||Bu||^2 + \frac{1}{4\delta}\left(\int_0^t g^{2-p}(\tau)d\tau\right)(g^p \circ Bu)(t)$$
.

Theorem 2.8 *Let* $(u_0, u_1) \in V \times H$ *be given. Assume that* g *satisfies* (3) *and* (4)*. Then, for each* $t_0 > 0$*, there exist strictly positive constants* K *and* k *such that the solution of* (1) *satisfies, for all* $t \geq t_0$*,*

$$\begin{aligned} \mathcal{E}(t) &\leq Ke^{-kt}, \qquad p = 1 \\ \mathcal{E}(t) &\leq K(1+t)^{-1/(p-1)}, \quad p > 1. \end{aligned} \tag{27}$$

Proof

Since g is positive, continuous, and $g(0) > 0$ then for any $t_0 > 0$ we have

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0, \; \forall t \geq t_0. \tag{28}$$

Use of above lemmas $\Longrightarrow$

$$L'(t) \leq -[\varepsilon_2\{g_0 - \delta\} - \varepsilon_1] ||u_t||^2 - \varepsilon_1 F(u)$$

$$
-\left[\frac{\varepsilon_1 l}{2} - \varepsilon_2\delta\{1+2(1-l)^2 + C_p^2(\frac{\mathcal{E}(0)}{l})^{\alpha-1}\}\right]||Bu||^2 \tag{29}
$$

$$
-\xi\left(\frac{1}{2} - \varepsilon_2\frac{g(0)}{4\delta}C_p - [\frac{\varepsilon_1}{l} + \varepsilon_2\{2\delta + \frac{1}{\delta}\}] \int_0^t g^{2-p}(\tau)d\tau\right)(g^p \circ Bu)(t)
$$

At this point we choose δ so small that

$$
g_0 - \delta > \tfrac{1}{2}g_0
$$

$$
\tfrac{2}{l}\delta\{1+2(1-l)^2 + C_p^2(\tfrac{\mathcal{E}(0)}{l})^{\alpha-1}\} < \tfrac{1}{4}g_0.
$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$
\frac{1}{4}g_0\varepsilon_2 < \varepsilon_1 < \frac{1}{2}g_0\varepsilon_2 \tag{30}
$$

will make

$$
\begin{aligned}
k_1 &= \varepsilon_2\{g_0 - \delta\} - \varepsilon_1 > 0 \\
k_2 &= \frac{\varepsilon_1 l}{2} - \varepsilon_2\delta\{1+2(1-l)^2 + C_p^2(\frac{\mathcal{E}(0)}{l})^{\alpha-1}\} > 0.
\end{aligned}
$$

We then pick ε_1 and ε_2 so small that (11) and (30) remain valid and

$$\left(\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_p - [\frac{\varepsilon_1}{l} + \varepsilon_2\{2\delta + \frac{1}{\delta}\}] \int_0^\infty g^{2-p}(\tau)d\tau\right) > 0$$

Therefore, for all $t \geq t_0$. we have

$$L'(t) \ \leq \ -\beta \left[||u_t||^2 + ||Bu||^2 + (g^p \circ Bu)(t) + F(u)\right] \tag{31}$$

Case 1. $p = 1$: We combine (6), (14) and (30) to get

$$L'(t) \ \leq \ -\beta_1 \mathcal{E}(t) \leq -\beta_1 \alpha_1 L(t) \qquad \forall t \geq t_0 \tag{32}$$

A simple integration of (32) leads to

$$L(t) \ \leq \ L(t_0) e^{\beta_1 \alpha_1 t_0} e^{-\beta \alpha_1 t}, \qquad \forall t \geq t_0. \tag{33}$$

which implies

$$\mathcal{E}(t) \leq \alpha_2 L(t_0) e^{\beta \alpha_1 t_0} e^{-\beta \alpha_1 t} = K e^{-kt}, \ \forall t \geq t_0. \tag{34}$$

Case 2. $p > 1$:

Conditions (3) and (4) $\Longrightarrow$

$$\int_0^\infty g^{1-\theta}(\tau)d\tau < \infty, \quad \theta < 2 - p,$$

so lemma 2.2 yields

$$\begin{aligned}(g \circ Bu)(t) \ &\leq \ C\left\{(g^p \circ Bu)(t)\right\}^{\theta/(p-1+\theta)} \\ &\times \left\{\left(\int_0^\infty g^{1-\theta}(\tau)d\tau\right)\mathcal{E}(0)\right\}^{(p-1)/(p-1+\theta)}\end{aligned} \tag{35}$$

Therefore we get, for $\sigma > 1$,

$$\begin{aligned}\mathcal{E}^\sigma(t) \ &\leq \ C\mathcal{E}^{\sigma-1}(0)\left(||u_t||^2 + ||Bu||^2 + F(u)\right) \\ &\quad +C\left\{(g \circ Bu)(t)\right\}^\sigma \\ &\leq \ C\mathcal{E}^{\sigma-1}(0)\left(||u_t||^2 + ||Bu||^2 + F(u)\right) \\ &\quad +C\left\{\mathcal{E}(0)\int_0^\infty g^{1-\theta}(\tau)d\tau\right\}^{\sigma(p-1)/(p-1+\theta)} \\ &\quad \times\left\{(g^p \circ Bu)(t)\right\}^{\sigma\theta/(p-1+\theta)},\end{aligned} \tag{36}$$

where C is a generic positive constant. By choosing $\theta = \frac{1}{2}$ and $\sigma = 2p - 1$ (hence $\sigma\theta/(p - 1 + \theta) = 1$), estimate (36) gives

$$\mathcal{E}^{\sigma}(t) \leq C\left\{||u_t||^2 + ||Bu||_2^2 + (g^p \circ Bu)(t) + F(u)\right\} \tag{37}$$

Combining (14), (31) and (37), we obtain

$$L'(t) \leq -\beta_2 \mathcal{E}^{\sigma}(t) \leq -\beta_2 \left(\alpha_1\right)^{\sigma} L^{\sigma}(t), \ \forall t \geq t_0, \tag{38}$$

where $\beta_2 > 0$. Integration of (38) gives

$$L(t) \leq C(1+t)^{-1/(\sigma-1)}, \qquad \forall t \geq t_0. \tag{39}$$

As a consequence of (39), we have

$$\int_0^{\infty} L(t)dt + \sup_{t \geq 0} tL(t) < \infty. \tag{40}$$

Using Lemma 3.3, we have

$$\begin{aligned} g \circ Bu \leq \\ C\left[\int_0^{\infty} \|Bu(s)\|^2 ds + \sup_t t\|Bu(t)\|^2\right]^{\frac{p-1}{P}} (g^p \circ Bu)^{1/p} \\ \leq \ C\left[\int_0^{\infty} L(s)ds + tL(t)\right]^{(p-1)/p} (g^p \circ \nabla u)^{1/p} \\ \leq \ C(g^p \circ \nabla u)^{1/p}. \end{aligned}$$

So

$$g^p \circ \nabla u \geq C(g \circ \nabla u)^p. \tag{41}$$

So (31) becomes

$$L^{'}(t) \leq -C\left[||u_t||^2 + \|Bu(t)\|^2 + (g \circ \nabla u)^p(t) + F(u)\right]. \tag{42}$$

Also,

$$\mathcal{E}^p(t) \leq C\left[||u_t||^2 + \|Bu(t)\|^2 + (g \circ \nabla u)^p(t) + F(u)\right]. \tag{43}$$

Combining the last two inequalities and (14), we obtain

$$L'(t) \leq -CL^p(t), \quad t \geq t_0. \tag{44}$$

A simple integration of (44) yields

$$L(t) \leq K(1+t)^{-1/(p-1)}, \quad t \geq t_0.$$

This completes the proof.

Remark . Estimates (27) also hold for all $t \in [0, t_0]$ by virtue of continuity and boundedness of $\mathcal{E}$.

3 Applications:

1)**Vector-valued Equation**

$$\begin{cases} u_{tt}(t) + Au(t) - \int\limits_0^t g(t-\tau)Au(\tau)d\tau = 0, \text{ in } (0,\infty) \\ u(0) = u_0 \in \mathbb{R}^n, \ u_t(0) = u_1 \in \mathbb{R}^n. \end{cases} \quad (45)$$

where $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a symmetric positive definite matrix

$u : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ is a one-variable vector-valued function

It is easy to verify that there exists a nonsingular symmetric matrix $B : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$A = B^2$$

Define

$$\mathcal{E}(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)|Bu(t)|^2 + \frac{1}{2} g \circ Bu(t)$$

where

$$g \circ Bu(t) = \int_0^t g(t-s)|Bu(t) - Bu(s)|^2 ds$$

Theorem 3.1: *Under the condtions (3) and (4) on* g, *the solution of* (45) *satisfies*

$$\mathcal{E}(t) \leq Ke^{-kt}, \qquad p = 1$$

$$\mathcal{E}(t) \ \leq \ K(1+t)^{-1/(p-1)}, \quad p > 1.$$

Proof. Take

$$H = V = \mathbb{R}^n$$

equipped with the norm $|\ |$ and use

$$Av.w = Bv.Bw, \quad \forall v, w \in \mathbb{R}^n$$

Also,

$$|u| = |B^{-1}Bu| \leq C_p|Bu|$$

2) **Wave Equation:**

$$\begin{cases} u_{tt} - \Delta u + \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau \\ +|u|^{\beta-2}u = 0, \text{ in } \Omega \times (0,\infty) \\ u(x,t) = 0, \ x \in \partial\Omega \, , t \geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{46}$$

$\Omega \subset \mathbb{R}^n$ ($n \geq 1$) bounded with $\partial\Omega$ regular, $g \geq 0$ satisfying (3) and (4), and

$$\begin{array}{ll} 2 \leq \beta \leq 2(n-1)/(n-2), & n > 2 \\ \beta \geq 2, & n = 1,2 \end{array}$$

In this problem, we take

$$H = L^2(\Omega), \quad V = H_0^1(\Omega), \quad A = -\Delta$$

It is well known that

$$< -\Delta u, v >= \int_\Omega \nabla u.\nabla v dx, \qquad \forall u, v \in V$$

and, by Poincare, we have

$$\int_\Omega u^2 dx \leq C_p \int_\Omega |\nabla u|^2 dx$$

Also,

$$f(u) = |u|^{\beta-2}u \quad and \quad F(u) = \frac{1}{\beta}\int_{\Omega}|u|^{\beta}$$

which gives

$$F(u(t)) - < f(u(t)), u >= (\frac{1}{\beta} - 1)\int_{\Omega}|u|^{\beta} \leq 0$$

The energy is

$$\begin{aligned}\mathcal{E}(t) &:= \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)||\nabla u(t)||^2 \\ &+ \frac{1}{2}||u_t||^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{\beta}\int_{\Omega}|u|^{\beta}dx.\end{aligned}$$

Theorem 3.2: *Under the condtions (3) and (4) on* g*, the solution of* (46) *satisfies*

$$\begin{aligned}\mathcal{E}(t) &\leq Ke^{-kt}, \qquad p = 1 \\ \mathcal{E}(t) &\leq K(1+t)^{-1/(p-1)}, \quad p > 1.\end{aligned}$$

.