1 Sobolev Spaces in $\mathbb{R}^N$

Definition: Let Ω be an open domain of $\mathbb{R}^N$ and $1 \leq p \leq +\infty$. We define the Sobolev space

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \quad / \quad \exists \quad g_1, g_2, \dots g_N \in L^p(\Omega)$$

satisfying

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} = - \int_\Omega g_i \, \phi, \quad \forall \, \phi \in C_0^\infty(\Omega) \Big\}$$

we denote by $H^1(\Omega) = W^{1,2}(\Omega)$. We have, in this case, $\nabla u = (g_1, \dots, g_N)$.
We equip $W^{1,p}(\Omega)$ with the norm

$$||u||_{1,p} = ||u||_p + \sum_{i=1}^N \left|\left| \frac{\partial u}{\partial x_i} \right|\right|_p$$

or equivalently with

$$||u||_{1,p} = \left(||u||_p^p + \sum_{i=1}^N \left|\left| \frac{\partial u}{\partial x_i} \right|\right|_p^p \right)^{\frac{1}{p}}, \quad 1 < p < \infty.$$

It is also clear that $H^1(\Omega)$ equipped with scalar product

$$\langle u_1 v \rangle = \langle u_1 v \rangle_{L^2} + \sum_{i=1}^N \langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \rangle_{L^2}$$

is a inner product space. The associated norm, in this case, is

$$||u||_{1,2} = \left(||u||_{L^2}^2 + \sum_{i=1}^N \left|\left| \frac{\partial u}{\partial x_i} \right|\right|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Theorem: *$W^{1,p}(\Omega)$ equipped with the above norm is a Banach space.*
Proof: Given a Cauchy sequence (u_n) in $W^{1,p}(\Omega) \Rightarrow (u_n)$ is Cauchy in $L^p(\Omega)$ and $\left(\frac{\partial u_n}{\partial x_i}\right)$ is Cauchy in $L^p(\Omega)$ for $1 \leq i \leq N$. So $u_n \longrightarrow u$ in $L^p(\Omega)$ and $\frac{\partial u_n}{\partial x_i} \longrightarrow g_i$ in $L^p(\Omega)$.
We have

$$\int_\Omega u_n \frac{\partial \phi}{\partial x_i} = - \int_\Omega \frac{\partial u_n}{\partial x_i} \, \phi, \quad \forall \, \phi \in C_0^\infty(\Omega)$$

By letting n go to ∞, we get

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} = - \int_\Omega g_i \, \phi, \quad \forall \, 1 \leq i \leq N$$

Therefore $u \in W^{1,p}(\Omega)$ with

$$\frac{\partial u}{\partial x_i} = g_i, \quad \forall\, 1 \le i \le N.$$

This shows that $W^{1,p}(\Omega)$ is a Banach space.
Corollary: $H^1(\Omega)$,*equipped with the above scalar product, is a Hilbert space.*
Theorem: $W^{1,p}(\Omega), 1 < p < \infty$ *is reflexive and* $W^{1,p}(\Omega), 1 \le p < \infty$ *is separable.*

For the proof, we only repeat the argument of the one-dimensional case, taking the operator

$$T : W^{1,p}(\Omega) \to \left[L^p(\Omega)\right]^{N+1}$$
$$u \to T(u) = (u, \nabla u).$$

Exercise: Suppose $u_n \to u$ in $L^p(\Omega)$ and (∇u_n) remains bounded in $\left[L^p(\Omega)\right]^N$. Then $u \in W^{1,p}(\Omega)$.
Example: Let $\Omega = (-1,1) \times (-1,1)$ be an open of $\mathbb{R}^2$ and $\omega = (0,1) \times (0,1) \subset \Omega$. We set $u = xy \;\; \chi_w$, where χ_w is the characteristic of ω. So for $\phi \in C_0^1(\Omega)$,
we have

$$\int\int_\Omega u\phi_x \, dx\, dy = \int_0^1 \int_0^1 u \;\; \phi_x \, dx\, dy$$

$$= \int_0^1 y \left(\int_0^1 x\, \phi_x \, dx \right) dy = \int_0^1 y \left[x\,\phi|_{x=0}^{x=1} - \int_0^1 \phi \, dx \right] dy$$

$$= -\int_0^1 \int_0^1 y\, \phi(x,y) dx\, dy = -\int\int_\Omega y\, \chi_\omega\, \phi \, dx\, dy$$

Similarly we have

$$\int\int_\Omega u\phi_y \, dx\, dy = -\int\int_\Omega x\, \chi_\omega\, \phi \, dx\, dy$$

Thus

$$u \in W^{1,P}(\Omega), \quad \forall\, 1 \le p \le +\infty,$$

with $\nabla_u = (y\psi_\omega,\, x\psi_\omega)$.
Example: Let

$$u(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

be defined on $\Omega = \{(x,y) | x^2 + y^2 < 1\}$.

It is easy to see that $u \in L^\infty(\Omega)$ so it is in $L^p(\Omega) \quad \forall p \ge 1$. Next, for $\phi \in C_0^1(\Omega)$ we have

$$
\begin{aligned}
\int\int_\Omega u\,\phi_x\,dx\,dy &= \int_0^1\int_0^{2\pi} u\left(\phi_r\,\cos\theta - \phi_\theta\frac{\sin\theta}{r}\right) r dr d\theta \\
&= \int_0^{2\pi}\cos\theta\int_0^1 ur\,\phi_r\,dr\,d\theta - \int_0^1\int_0^{2\pi}(u\,\sin\theta)\phi_\theta\,d\theta\,dr \\
&= -\int_0^{2\pi}\cos\theta\int_0^1\left[(r\,u_r\,\phi + u\phi)dr\right]\,d\theta \\
&\quad + \int_0^1\int_0^{2\pi}\phi\,(u_\theta\sin\theta + u\,\cos\theta)\,d\theta dr \\
&= -\int_0^{2\pi}\int_0^1 r\,u_r\,\phi\,\cos\theta\,dr\,d\theta + \int_0^1\int_0^{2\pi}\phi u_\theta\sin\theta\,d\theta\,dr \\
&= -\int_0^{2\pi}\int_0^1\left(u_r\;\cos\theta - u_\theta\frac{\sin\theta}{r}\right)\phi\,r dr\,d\theta = \int\int_\Omega u_x\,\phi dx\,dy
\end{aligned}
$$

We conclude that

$$
\nabla_u = \left(\frac{\partial u}{dx}, \frac{\partial u}{\partial y}\right) = \left(\frac{y(y^2-x^2)}{(x^2+y^2)^2}, \;\frac{x(x^2-y^2)}{(x^2+y^2)^2}\right)
$$

Let's verify that $\nabla u \in \left[L^p(\Omega)\right]^2$:

$$
\begin{aligned}
\int\int_\Omega\left|\frac{\partial u}{\partial x}\right|^p dx\,dy &= \int_0^{2\pi}\int_0^1\frac{r^{3p}\left|\sin\theta(\sin^2\theta-\cos^2\theta)\right|^p}{r^{4p}}\,r\,dr\,d\theta \\
&\le \int_0^{2\pi}\int_0^1 r^{1-p}dr\,d\theta < \infty
\end{aligned}
$$

iff $p<2$. Similarly we estimate $\int\int_\Omega\left|\frac{\partial u}{\partial y}\right| dx\,dy$. Therefore $u \in W^{1,p}(\Omega), 1\le p<2$.
Remark: This example shows that functions in $W^{1,p}(\Omega)$ are not necessarily continuous if the space dimension $N\ge 2$.
Theorem: *Let* $u\in W^{1,p}(\Omega)$. *Then*
1)

$$
\tilde{u} = \begin{cases} u & \text{on} \quad \Omega \\ 0 & \text{on} \quad \mathbb{R}^N\backslash\Omega \end{cases} \quad \in L^P\left(\mathbb{R}^N\right)
$$

2) *If* $\alpha\in C_0^1(\Omega)$ then $\widetilde{\alpha u}\in W^{1,p}\left(\mathbb{R}^N\right)$ *and*

$$
\frac{\partial}{\partial x_i}(\widetilde{\alpha u}) = \widetilde{\frac{\partial\alpha}{\partial x_i}}u + \alpha\widetilde{\frac{\partial u}{\partial x_i}}.
$$

Proof.
1) is trivial
2)

$$
\int_{\mathbb{R}^N}\widetilde{\alpha u}\;\;\phi_{x_i} \;=\; \int_\Omega \alpha u\,\phi_{x_i}, \quad \forall\,\phi\in C_0^1\left(\mathbb{R}^N\right)
$$

$$\begin{aligned} &= \int_\Omega u\left[(\alpha\phi)_{x_i} - \phi\alpha_{x_i}\right] \\ &= -\int_\Omega \left(u_{x_i}\alpha\phi + u\alpha_{x_i}\phi\right) = -\int_\Omega \phi\left(u_{x_i}\alpha + u\alpha_{x_i}\right) \\ &= -\int_{\mathbb{R}} \phi\left(\alpha\widetilde{u_{x_i}} + u\widetilde{\alpha_{x_i}}\right) \end{aligned}$$

Since $\alpha\widetilde{\frac{\partial u}{\partial x_i}} + \widetilde{\frac{\partial \alpha}{\partial x_i}} \in L^p\left(\mathbb{R}^N\right)$ then $\alpha\tilde{u} \in W^{1,p}\left(\mathbb{R}^N\right)$ with

$$\frac{\partial}{\partial x_i}(\widetilde{\alpha u}) = \alpha\widetilde{\frac{\partial u}{\partial x_i}} + u\widetilde{\frac{\partial \alpha}{\partial x_i}}.$$

Lemma: *Let $\rho \in L^1(\mathbb{R}^N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. Then $\rho * u \in W^{1,p}(\mathbb{R}^N)$ and*

$$\frac{\partial}{\partial x_i}(\rho * u) = \rho * \frac{\partial u}{\partial x_i}.$$

The proof of this lemma goes exactly like the one in the one-dimensional case.
Theorem: (Friedrichs)

Suppose that $u \in W^{1,p}(\Omega), 1 \leq p < \infty$, then there exists a sequence $(u_n) \subset C_0^\infty(\mathbb{R}^N)$ such that

1)

$$u_{n|\Omega} \longrightarrow u \quad in \quad L^p(\Omega)$$

2)

$$\frac{\partial u_n}{\partial x_i}|_\omega \rightarrow \frac{\partial u}{\partial x_i}, 1 \leq i \leq N \quad in \quad L^p(\omega), \quad \forall w \subset\subset \Omega.$$

Proof. Let

$$\tilde{u} = \begin{cases} u & \text{on } \Omega \\ 0 & \text{on } \mathbb{R}^N \backslash \Omega \end{cases}$$

and

$$\rho \in C_0^\infty(\mathbb{R}^N), \quad \rho \geq 0, \quad supp\rho \subset\subset B(0,1), \quad \int_{\mathbb{R}^N} \rho = 1.$$

We define a regularizing sequence $\rho_n(x) = n^N \rho(nx)$. It is clear that

$$supp\rho_n \subset B\left(0, \frac{1}{n}\right) \text{ and } \int_{\mathbb{R}^N} \rho_n(x) = 1, \quad \forall n = 1, 2, 3, ...$$

We then set $v_n = \rho_n * \tilde{u}$. So $v_n \in\subset^\infty (\mathbb{R}^N)$ and $v_n \longrightarrow \tilde{u}$ in $L^p(\mathbb{R}^N)$.

For $w \subset\subset \Omega$, let $\alpha \in\subset_0^1 (\Omega)$ such that $0 \leq \alpha \leq 1$ and $\alpha \equiv 1$ on a neighborhood $\bigcup$, of w, contained in Ω.

Also for n so large, we have

$$\rho_n * \widetilde{\alpha u}\,|_\omega = \rho_n * \tilde{u}|_\omega\,, \qquad (i)$$

since

$$supp\left(\rho_n * \widetilde{\alpha u} - \rho_n * \tilde{u}\right) = supp\left(\rho_n * (\widetilde{\alpha u} - \tilde{u})\right)$$

$$\subset \left(B\left(0,\frac{1}{n}\right)+\Omega\backslash\bigcup\right) \subset \omega^c,$$

for n large. From the above lemma, we have

$$\frac{\partial}{\partial x_i}\left(\rho_n * \widetilde{\alpha u}\right) = \rho_n * \left(\alpha\widetilde{\frac{\partial u}{\partial x_i}} + u\widetilde{\frac{\partial \alpha}{\partial x_i}}\right)$$

therefore

$$\frac{\partial}{\partial x_i}\left(\rho_n * \widetilde{\alpha u}\right) \longrightarrow \alpha\widetilde{\frac{\partial u}{\partial x_i}} + u\widetilde{\frac{\partial \alpha}{\partial x_i}} \text{ in } L^p(\mathbb{R})^N.$$

In particular

$$\frac{\partial}{\partial x_i}\left(\rho_n * \widetilde{\alpha u}\right) \longrightarrow \frac{\partial u}{\partial x_i} \text{ in } L^p(\omega).$$

Hence, by (i), we see that

$$\frac{\partial}{\partial x_i}\left(\rho_n * \tilde{u}\right) \longrightarrow \frac{\partial u}{\partial x_i} \text{ in } L^p(\omega)$$

Conclusion $v_n \in \subset^\infty (\mathbb{R}^N)$ and

$$v_n\,|_\Omega \to u \text{ in } L^p(\Omega)$$

$$\frac{\partial}{\partial x_i}v_n \longrightarrow \frac{\partial u}{\partial x_i} \text{ in } L^p(\omega), \quad w \subset\subset \Omega, \quad 1 \leq i \leq N.$$

Finally we put $u_n = \xi_n v_n$, where ξ_n is a truncation sequence. We then easily verify that u_n has the properties required by the theorem.
Remark: In general, for $u \in W^{1,p}(\Omega)$ we cannot always find a sequence (u_n) in $C_0^\infty(\mathbb{R}^N)$ such that $u_{n|\Omega} \longrightarrow u$ in $W^{1,p}(\Omega)$.