1 Extension and Density

Definition: Given a function $f : I \to \mathbb{R}$ and J an interval such that $I \subset J$. We say that $\tilde{f}$ is an extension of f on J if $\tilde{f}(x) = f(x), \quad \forall\, x \in I$.

Theorem: (Extension Operator): Given $1 \leq p \leq \infty$. There exists an operator $P : W^{1,p} \to W^{1,p}(\mathbb{R})$ such that

(i) $P_{u_{|_I}} = u, \quad \forall\, u \in W^{1,p}(I)$,

(ii) $\|P_u\|_p \leq C\|u\|_{1,p}, \quad \forall\, u \in W^{1,p}(I)$,

(iii) $\|P_u\|_{1,p} \leq C'\|u\|_{1,p}, \quad \forall\, u \in W^{1,p}(I)$.

Proof: Suppose that $I = (a, b)$ bounded. We can take $I = (0, 1)$ by a change of variable $t = a + s(b - a)$ Let η be a $C^1(\mathbb{R})$ function such that

$$\eta(x) = \begin{cases} 1, & x < 1/4 \\ 0, & x > 3/4 \end{cases}$$

We extend u to $(0, \infty)$ by $\eta\tilde{u}$ where

$$\tilde{u}(x) = \begin{cases} u(x), & x \in I \\ 0, & x \geq 1. \end{cases}$$

Let us show that $\eta\tilde{u} \in W^{1,p}((0, +\infty))$

$$\int_0^{+\infty} |\eta\tilde{u}|^p = \int_0^1 |\eta u|^p \leq \int_0^1 |u|^p \Rightarrow \|\eta\tilde{u}\|_{L^p(\mathbb{R})} \leq \|u\|_{L^p(\mathbb{R})}.$$

Let $\varphi \in C_0^\infty(\mathbb{R})$, then

$$\begin{aligned} \int_0^{+\infty} \eta\tilde{u}\varphi'(x)dx &= \int_0^1 \eta u\varphi' dx \\ &= \int_0^1 u[(\eta\varphi)' - \eta'\varphi]dx \\ &= \int_0^1 u(\eta\varphi)' - \int_0^1 u\eta'\varphi \\ &= -\int_0^1 u'\eta\varphi - \int u\eta'\varphi = -\int_0^1 (u'\eta + u\eta')\varphi. \end{aligned}$$

since $\eta\varphi \in C_0^1(I)$. Thus

$$\int_0^{+\infty} \eta\tilde{u}\varphi' = -\int_0^{+\infty} (\tilde{u}'\eta + \tilde{u}\eta')\varphi$$

where

$$\tilde{u}' = \begin{cases} u' & \text{on } I \\ 0 & \text{on } [1, +\infty) \end{cases}$$

and it is easy to see that

$$\begin{aligned} \|\eta\tilde{u}' + \eta'\tilde{u}\|_{L^p(\mathbb{R}^+)} &= \|\eta u' + \eta' u\|_{L^p(I)} \\ &\leq \|\eta u'\|_{L^p(I)} + \|\eta' u\|_{L^p(I)} \\ &\leq \|u'\|_{L^p(I)} + \|\eta'\|_\infty \|u\|_{L^p(I)} \end{aligned}$$

So

$$\|\eta\tilde{u}\|_{1,p} \leq C\|u\|_{1,p}.$$

Now we are in the first case and hence $\eta\tilde{u}$ is extended to $\mathbb{R}$ by reflexion. We denote by v_1 extension of ηu to $\mathbb{R}$.
Similarly, we do for $(1-\eta)u$ and we denote by v_2 the extension of $(1-\eta)u$ to $\mathbb{R}$.

Let

$$v = v_1 + v_2;$$

so $v \in W^{1,p}(\mathbb{R})$ and $v(x) = u(x)$, $\forall\, x \in I$. This completes the proof.
Remark: If $u \in W^{1,p}(I),\ I = (a,b)$ extending u by zero outside I, will not give in general $\tilde{u}$ in $W^{1,p}(\mathbb{R})$.
Example: Let $u(x) = x$ on $(0,1)$. So

$$\tilde{u}(x) = \begin{cases} x, & x \in (0,1) \\ 0, & x \notin (0,1) \end{cases}$$

Let $\varphi \in C_0^\infty(\mathbb{R})$ so

$$\begin{aligned} \int_{-\infty}^{\infty} \tilde{u}\varphi'(x)dx &= \int_0^1 x\varphi'(x)dx \\ &= x\varphi(x)|_0^1 - \int_0^1 \varphi(x)dx \\ &= \varphi(1) - \int_0^1 \varphi(x)dx \\ &= \delta_1\varphi - \int_0^1 \varphi = \delta_1\varphi - \int_{-\infty}^{\infty} \tilde{u}'\varphi. \end{aligned}$$

Hence $\tilde{u} \notin W^{1,p}(\mathbb{R})$.
Remark: The extension operator is not unique. It depends heavily on the choice of η.
Definition: (Convolution). Let ρ be in $L^1(\mathbb{R})$ and g in $L^p(\mathbb{R})$. We define the convolution of ρ and g by

$$\rho * g(x) = \int_{-\infty}^{\infty} \rho(x-y)g(y)dy, \quad \text{for almost every } x \in \mathbb{R}$$

Jension Inequality. If $f \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$ and $\phi \geq 0$ continuous and convex, then

$$\phi\left(\int fg\right) \leq \int f\phi(g),$$

Properties:
1) $\rho * g = g * \rho$.
2) If $\rho \in L^1$ and $g \in L^p$, then $\rho * g$ is in $L^p(\mathbb{R})$. Moreover

$$\|\rho * g\|_{L^p} \leq \|\rho\|_{L^1} \cdot \|g\|_{L^p}.$$

3) If $\rho \in L^1(\mathbb{R}),\ g \in C^0(\mathbb{R})$, then $\rho * g \in L^\infty(\mathbb{R})$ Moreover we have

$$\tau_h(\rho * g) - \rho * g = \rho * (\tau_h g - g)$$

and

$$\|\rho * (\tau_n g - g)\|_{L^\infty} \leq \|\rho\|_{L^1} \|\tau_h g - g\|_{L^\infty}.$$

Hence

$$\|\rho * (\tau_h g - g)\|_{L^\infty} \to 0 \text{ as } h \to 0.$$

4) If $f \in L^1(\mathbb{R})$ and $\varphi \in C_0^1(\mathbb{R})$ or $C_0^\infty(\mathbb{R})$, then $f*\varphi \in C^1$ or C^∞ with $(f*\varphi)' = f*\varphi'$.
5) If f and φ are of compact support, then $f * \varphi$ is of compact support too.
Proof: 1)

$$\int_{-\infty}^{\infty} \rho(x-y)g(y)dy = -\int_{\infty}^{-\infty} \rho(s)g(x-s)ds,$$

using the change of variables $s = x - y \Rightarrow dy = -ds$.
2) let

$$f = \frac{|\rho|}{\|\rho\|_{L^1}} \phi(r) = r^p, \quad p \geq 1$$

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{\rho(y)}{\|\rho\|_{L^1}} g(x-y)dy \right|^p dx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\rho(y)|}{\|\rho\|_{L^1}} |g(x-y)|^p dydx$$

that is

$$\frac{1}{\|\rho\|_{L_1}^p} \|g * \rho\|_{L^1}^p \leq \int_{-\infty}^{\infty} \frac{|\rho(y)|}{\|\rho\|_{L^1}} dy \int_{-\infty}^{\infty} |g(x-y)|^p dx = \|g\|_{L^p}^p$$

thus

$$\|g * \rho\|_{L^1} \leq \|\rho\|_{L_1} \|g\|_{L^p}$$

3)

$$|\rho * g(x)| \leq \int_{-\infty}^{\infty} \rho(x-y)|\ |g(y)|dy = \|g\|_{L^\infty} \cdot |\rho\|_{L^1}$$

hence $\rho * g \in L^\infty$.

$$\begin{aligned}
\tau_h(\rho * g)(x) - g * p(x) &= \int_{-\infty}^{\infty} \rho(y)g(x+h-y)dy - \int_R \rho(y)g(x-y)dy \\
&= \int_{-\infty}^{\infty} \rho(y)[g(x+h-y) - g(x-y)]dy \\
&= \int_{-\infty}^{\infty} \rho(y)[\tau_h g(x-y) - g(x-y)]dy \\
&= \rho * (\tau_h g - g)(x).
\end{aligned}$$

It is also clear that

$$\|\rho * (\tau_h g - g)\|_{L^\infty} \leq \|\rho\|_{L^1} \|\tau_h g - g\|_{L^\infty}$$

and hence g is continuous, then $\|\tau_h g - g\|_{L^\infty} \to 0$ as $h \to 0$.
4)

$$\begin{aligned} & \frac{(f*\varphi)(x+h)-(f*\varphi)(x)}{h}-(f*\varphi')(x) \\ = & \int_{-\infty}^{\infty} f(y)\left[\frac{\varphi(x+h-y)-\varphi(x-y)}{h}-\varphi'(x-y)\right]dy. \end{aligned}$$

and since $\varphi \in C_0^1(\mathbb{R})$, then we take limit to get

$$\lim_{h\to\infty} \frac{(f*\varphi)(x+h)-(f*\varphi)(x)}{h}-(f*\varphi')(x)=0$$

hence $(f*\varphi)'(x) = f*\varphi'(x)$.

Now, similarly if $\varphi \in C_0^2(\mathbb{R})$, then

$$(f*\varphi)'' = (f*\varphi')' = f*\varphi''$$

and we continue to find that

$$(f*\varphi)^{(m)} = f*\varphi^{(m)}, \quad \forall\, m=1,2,3,\ldots.$$

5) If support $f \subset (-r,r)$ and support $\varphi \subset (R,R)$, then supp $f*\varphi \subset (-r-R, r+R)$. Let $x > r+r$, so

$$f*\varphi(x) = \int_{-r}^{r} f(y)\varphi(x-y)dy$$

If $x > r+R \Rightarrow x-y > r+R-r = R \Rightarrow \varphi(x-y) \Longrightarrow f*\varphi(x) = 0$.

Similarly, if $x < -r-R$, we have $f*\varphi(x)=0$.

Lemma: Suppose that $\rho \in L^1(\mathbb{R})$ and $v \in W^{1,p}(\mathbb{R})$. then $(\rho * v)$ is in $W^{1,p}(\mathbb{R})$ with $(\rho*v)' = \rho * v'$.

Proof: First suppose that ρ is of compact support and let $\varphi \in C_0^1(\mathbb{R})$

$$\begin{aligned} \int_{-\infty}^{\infty}(\rho*v)\varphi'(x) &= \int_{-k}^{k}\int_{-\infty}^{\infty}\rho(y)v(x-y)\varphi'(x)dydx. \\ &= \int_{-k}^{k}\int_{-\infty}^{\infty} v(y)\rho(x-y)\varphi'(x)dydx \\ &= \int(\int(\rho(x-y)\varphi'(x)dx)v(y)dy \\ &= \int_{-\infty}^{\infty}(\check{\rho}*\varphi)'(y)v(y)dy = -\int v'(\check{\rho}*\varphi)(y)dy \\ &= -\int v'(y)\int\rho(x-y)\varphi(x)dxdy = -\int(\int\rho(x-y)\varphi(x)v'(y)dx)dy \\ &= -\int\varphi(x)\left(\int\rho(x-y)v'(y)dy\right)dx \\ &= -\int(\rho*v')(x)\varphi(x)dx, \quad \forall\,\varphi\in C_0^1(\mathbb{R}). \end{aligned}$$

Here $\check{\rho}(s) = \rho(-s)$. Moreover we have

$$\check{\rho}*\varphi \in C_0^1(\mathbb{R}) \text{ since } v \in W^{1,p}(\mathbb{R})$$

We conclude then

$$\rho * v \in W^{1,p} \text{ with } (\rho * v)' = \rho * v'.$$

Now, for $\rho \in L^1(\mathbb{R})$ there exists a sequence $(\rho_n) \subset C_0^\infty(\mathbb{R})$ such that $\rho_n \to \rho$ in L^1 norm. $\rho_n * v$ is in $W^{1,p}(\mathbb{R})$ such that $(\rho_n * v)' = \rho_n * v'$.

$$\|\rho_n * v - \rho * v\|_{L^p} = \|(\rho_n - \rho) * v\|_{L^p} \leq \|\rho_n - \rho\|_{L^1} \|v\|_{L^p}$$

so as $n \to \infty$, we get $\rho_n * v \to \rho * v$ in L^p. Similarly, we get

$$\rho_n * v' \to \rho * v' \text{ in } L^p.$$

Therefore $\rho * v$ is in $W^{1,p}(\mathbb{R})$ with $(\rho * v)' = \rho * v'$.
Truncation. Let ξ be a $C_0^\infty(\mathbb{R})$ function with

$$\xi(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| \geq 2 \end{cases}$$

We define

$$\xi_n(x) = \xi\left(\frac{x}{n}\right), \quad n = 1, 2, 3, ...$$

Lemma: Let f be in $L^p(\mathbb{R}), 1 \leq p < \infty$, Then $\xi_n f \to f$ in L^p.
Proof: $|\xi_n f - f|^p \to 0$, almost everywhere on $\mathbb{R}$.

$$|\xi_n f - f|^p \leq 2^{p-1}\left(|\xi_n f|^p + |f|^p\right) \leq 2^p |f|^p \in L^1(\mathbb{R})$$

So by the dominated convergence theorem we get

$$\int_{-\infty}^{\infty} |\xi_n f - f|^p \to 0.$$

Modifiers: Let

$$\psi(x) = \begin{cases} \exp\left[-1/(x-1)^2\right], & -1 < x < 1 \\ 0, & |x| \geq 1 \end{cases}$$

It is easy to check that $\psi \in C_0^\infty(\mathbb{R})$. We define the molifiers or the regularizing sequences by

$$\psi_n(x) = \frac{n}{a}\psi(nx), \qquad a = \int_{-\infty}^{\infty} \psi(x)dx$$

Properties
1) $\int_{-\infty}^{\infty} \psi_n(x)dx = 1$.
2) $\psi_n \to \delta$ in $\mathcal{D}'(\mathbb{R})$ with $\text{supp}\psi_n \subset (-1/n, 1/n)$, hence $\text{supp}\psi_n \to \{0\}$.
Proof: 1) Direct integration
2) For $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned} |\int_{-\infty}^{\infty} \psi_n(x)\varphi(x)dx - \varphi(0)| &= |\int_{-\infty}^{\infty} \psi_n(x)\left[\varphi(x) - \varphi(0)\right]dx| \\ &= |\frac{1}{a}\int_{-\infty}^{\infty} \rho(y)\left[\varphi(\frac{y}{n}) - \varphi(0)\right]dy| \\ &\leq \frac{1}{a}\max_{[-1,1]}\left[\varphi(\frac{\zeta}{n}) - \varphi(0)\right]\int_{-\infty}^{\infty} \rho(y)dy \\ &\leq \frac{1}{a}\max_{[-1,1]}\left[\varphi(\frac{\zeta}{n}) - \varphi(0)\right] \to 0 \end{aligned}$$

as n $\to \infty$ since φ is uniformly continuous on $[-1,1]$. Hence $\psi_n \to \delta$ in $\mathcal{D}'(\mathbb{R})$.
Theorem (Density): Let u be in $W^{1,p}(I), 1 \leq p < \infty$,then there exitx a sequence $(u_n) \subset C_0^\infty(\mathbb{R})$, such that $u_{n|I} \to u$ in $W^{1,p}(I)$.
Proof: If $I \neq \mathbb{R}$ we then extend u to $\mathbb{R}$ by Pu. So let us suppose that $I = \mathbb{R}$. We take $u_n = \xi_n(\rho_n * u)$

$$\begin{aligned}
||u_n - u||_p &= ||\xi_n(\rho_n * u) - u||_p \\
&\leq ||\xi_n(\rho_n * u - u)||_p + ||\xi_n u - u||_p \\
&\leq ||\xi_n||_\infty ||\rho_n * u - u||_p + ||\xi_n u - u||_p \to 0
\end{aligned}$$

as $n \to \infty$ by the above lemma
$u_n' = \xi_n'(\rho_n * u) + \xi_n(\rho_n * u')$, so

$$\begin{aligned}
||u_n' - u'||_p &\leq ||\xi_n'(\rho_n * u)||_p + ||\xi_n(\rho_n * u') - u'||_p \\
&\leq \frac{C}{n}||u||_p + ||\xi_n(\rho_n * u') - u'||_p \to 0
\end{aligned}$$

as $n \to \infty$, by the same argument.