1 Sobolev Spaces

1.1 Motivation

Let us consider the problem

$$\left.\begin{array}{ll} -u''(x) + u(x) = f(x); & 0 < x < 1 \\ u(0) = u(1) = 0. & \end{array}\right\} \qquad (1)$$

The objective is to find a solution $u \in C^2([0,1])$ for $f \in C([0,1])$. This problem is solvable by standard calculus methods.

Suppose that $\varphi \in C_0^1((0,1))$; that φ is continuously differentiable and $\varphi(0) = \varphi(1) = 0$. So

$$\int_0^1 (-u'' + u)\varphi(x)dx = \int_0^1 f\varphi dx.$$

Using integration by parts, we obtain

$$\int_0^1 u'\varphi' + u\varphi = \int_0^1 f\varphi dx. \qquad (2)$$

We notice here that (2) is valid if u and $u' \in L^1((0,1))$ or "simply" $u \in C^1([0,1])$. In this case, we say that u is a weak solution of (1). It satisfies (1) in the weak or in the variational sense (2).

(2) is the variational equation for (1).

Theorem: If $f \in C^0$, then any weak solution $u \in C^1$ is C^2.

Proof: It suffices to note that $u'' = u - f \in C$. So $u \in C^2$.

Theorem: If $u \in C^1((0,1))$ with $u(0) = u(1) = 0$ satisfying (2) for all $\varphi \in C_0^1((0,1))$. Then u is a classical solution; that u satisfies (1).

Remark In some instance, we can show that a weak solution is in fact a classical solution. This is called the regularity theory. It will be a part of our course.

1.2 Sobolev Space $W^{1,p}(I)$

Definition: Given an open interval I and $1 \le p \le +\infty$ (bounded or not). We define the Sobolev space $W^{1,p}(I) = \{u \in L^p(I)$ such that there exists $g \in L^p(I)$, for which we have $\int_I u\varphi' = -\int_I g\varphi; \quad \forall \varphi \in C_0^1(I)\}$.

Remark : For $p = 2$, we denote $W^{1,2}(I) = H^1(I)$.

Remark : In such a case, we write $u' = g$ in the weak sense; that is

$$\int_I (u' - g)\varphi dx = 0; \quad \forall \varphi \in C_0^1(I).$$

Example: Let $I = (-1,1)$ and $u(x) = \frac{1}{2}(x + |x|)$. It is clear that u is not differentiable at $x = 0$ (hence on I). Let $\varphi \in C_0^1(I)$, then

$$\begin{aligned} \int_{-1}^1 u\varphi' &= \int_{-1}^0 0.\varphi' dx + \int_0^1 x\varphi' dx = \int_0^1 x\varphi' dx \\ &= x\varphi|_0^1 - \int_0^1 \varphi(x)dx = -\int_0^1 \varphi(x)dx = -\int_{-1}^1 g\varphi(x)dx \end{aligned}$$

where

$$g(x) = \begin{cases} 0; & -1 \leq x < 0 \\ 1; & 0 \leq x \leq 1. \end{cases}$$

This is the Heavyside function denoted by H. So $u' = H$ in the weak sense.
Note that $u, H \in L^p(I)$; $\forall p \geq 1 \Rightarrow u \in W^{1,p}(I)$; $\forall 1 \leq p \leq \infty$.
Now let us consider $H(x)$ and test if it is a $W^{1,p}(I)$ function. For this we take $\varphi \in C_0^1(I)$ as a test function

$$\int_{-1}^{1} H\varphi' dx = \int_0^1 \varphi'(x) = \varphi(1) - \varphi(0) = -\varphi(0) = -\delta(\varphi):$$

In this case, we cannot find $g \in L^p(I)$ such that

$$\int_{-1}^{1} g\varphi\, dx = -\varphi(0); \quad \forall \varphi \in C_0^1(I):$$

Remark: In this latter case, we say that $H' = \delta$ in the distributional sense. δ is a distribution. This also motivates us to define $W^{1,p}(I)$ by using distributions.
Definition: We say that $u \in W^{1,p}(I)$ if $u \in L^p(I)$ and its distribution derivative u' coincides with an $L^p(I)$ function g.
Theorem: The Sobolev space $W^{1,p}(I)$ equipped with the norm

$$\|W\|_{1;p} = \|u\|_p + \|u'\|_p$$

is a Banach space.
Proof: Let (u_n) be a Cauchy sequence in $W^{1,p}(I)$. So (u_n) and (u_n') are Cauchy in L^p $\Rightarrow$

$$u_n \to u \text{ and } u_n' \to g \text{ in } L^p(I):$$

Now

$$\int_I u_n\varphi' = -\int_I u_n'\varphi; \quad \forall n; \quad \varphi \in C_0^1(I):$$

As $n \to \infty$,

$$\int_I u\varphi' = -\int_I g\varphi; \quad \forall \varphi \in C_0^1(I)$$

$\Rightarrow u \in W^{1,p}(I)$ with $u' = g$ (Definition) and $\|u_n - u\|_{1;p} \to 0$ as $n \to \infty$.
Theorem: The space $W^{1,p}(I)$ is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$.
Proof: We define the operator

$$\begin{aligned} T : W^{1,p}(I) &\to L^p(I) \times L^p(I) \\ u &\to (u; u'): \end{aligned}$$

This is an isometry. So $T(W^{1,p}(I))$ is a closed subspace of $L^p(I) \times L^p(I)$; hence $T(W^{1,p}(I))$ is reflexive for $1 < p < \infty$ since $L^p \times L^p$ is reflexive for $1 < p < \infty$.
The same thing holds for separability.
Theorem: Given $u \in W^{1,p}(I)$. There exists $\tilde{u} \in C(\bar{I})$ such that $\tilde{u} = u$ a.e. in I and

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u'(t)dt; \quad \forall x; y \in I:$$

Proof: It is clear that

$$\tilde{u}(x) = u(x_0) + \int_{x_0}^{x} u'(t)dt; \quad x_0 \in I;$$

is absolutely continuous $\Rightarrow$ $\tilde{u} \in C(\bar{I})$. Moreover, $\tilde{u}' = u'$ a.e. in I $\Rightarrow$ $\tilde{u} = u + c$ a.e. in I. But $c = 0$ since $\tilde{u}(x_0) = u(x_0)$

Remark: This theorem shows that $W^{1,p}(I)$ functions can be represented by absolutely continuous functions defined on $\bar{I}$. This is why we usually say that if $u \in W^{1,p}(I)$ then $u \in C(\bar{I})$.

Remark: If u is absolutely continuous with $u' \in L^p(I)$, this does not mean that $u \in W^{1,p}(I)$ unless I is bounded.

Let

$$g(t) = \frac{1}{1+t^2} \in L^1($$

but

$$u(x) = \int_0^x g(t)dt = \tan^{-1} x \notin L^1($$

since

$$\int_0^R \tan^{-1} x dx = R\left[\tan^{-1} R - \frac{\log(1+R^2)}{2R}\right].$$

which gives

$$\int_0^\infty \tan^{-1} x dx = \infty.$$

Definition: Let u be defined in I; for x and h such that $x + h \in I$, we denote by

$$\tau_h f(x) = f(x+h)$$

.

Theorem: Let $u \in L^p(I)$; $1 < p \leq \infty$. The following properties are equivalent:

(i) $u \in W^{1,p}(I)$.

(ii) There exists a constant $C > 0$ such that

$$\left|\int_I u\varphi'\right| \leq C\|\varphi\|_{p'}; \quad \forall \varphi \in C^1_0(I); \quad \frac{1}{p'} + \frac{1}{p} = 1.$$

(iii) There exists a constant $C' > 0$ such that for each open $\omega \subset\subset I$ ($\bar{\omega} \subset I$) and for each h with $|h| <$ distance $(\omega, \partial I)$, we have

$$\|\tau_h u - u\|_p \leq C'|h|.$$

Proof: (i) $\Rightarrow$ (ii) by Holder's inequality with $c = \|u\|_p$.

(ii) $\Rightarrow$ (i) why?

We define $F : C(I) \to \mathbb{R}$ by

$$F(\varphi) = \int_I u\varphi'$$

which is bounded and continuous. By Hahn Banach theorem this form is extended to $L^{p'}(I)$. So

$$|F(\varphi)| \leq C\|\varphi\|_{L^{p'}}; \quad \forall \varphi \in L^{p'}.$$

By Riesz representation theorem, there exists $g \in L^p(I)$ such that
$$\int_I u\varphi' = \int_I g\varphi; \quad \forall \varphi \in L^{p'}(I);$$
in particular, we have
$$\int_I u\varphi' = \int_I g\varphi; \quad \forall \varphi \in C_0^1(I):$$
So, $u \in W^{1,p}(I)$ with $u' = -g$
(i) $\Rightarrow$ (iii) why ?
$$u(x+h) - u(x) = \int_x^{x+h} u'(t)dt:$$
Now for $p = +\infty$ we have
$$|u(x+h) - u(x)|_\infty \le \left|\int_x^{x+h} |u'|_\infty dt\right| \quad \|\tau_n u\|_\infty \le \|u\|_\infty |h|:$$
Next $1 < p < \infty$. We write
$$u(x+h) - u(x) = \int_0^1 h u'(x+sh)ds;$$
which implies
$$|u(x+h) - u(x)|^p \le |h|^p \int_0^1 |u'(x+sh)|^p ds$$
Integrationg over ω:
$$\int_\omega |u(x+h) - u(x)|^p dx \le |h|^p \int_\omega \int_0^1 |u'(x+sh)|^p ds dx$$
$$\le |h|^p \int_0^1 \|u'\|_p^p ds = |h|^p \|u'\|_p^p$$
Therefore
$$\|\tau_h u - u\|_p \le |h| \, \|u\|_{1;p}:$$
(iii) $\Rightarrow$ (ii) why?
Let $\varphi \in C_0^1(I) \Rightarrow \operatorname{supp}\varphi \subset \omega \subset\subset I$. Take h such that $|h| <$ distance $(\omega; \partial I)$, so
$$\int_I [u(x+h) - u(x)]\varphi(x)dx = \int_I [\varphi(x) - \varphi(x-h)]u(x)dx$$
This leads to
$$\left|\int_I [\varphi(x) - \varphi(x-h)]u(x)dx\right| \le \|\tau_h u\|_p \|\varphi\|_{p'} \le C|h| \|\varphi\|_{p'}$$
Divide by h and let $h \to 0$ to obtain
$$\left|\int_I u(x)\varphi'(x)dx\right| \le C\|\varphi\|_{p'}:$$

This completes the proof.
Remark: For $p = 1$, we only have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).
Remark: (ii) $\Rightarrow$ (i) is in general false because of the nonseparability of $L^\infty(I)$.