

# 1 Embedding

## Sobolev Inequalities

In the one-dimensional case, we saw that  $W^{1,p}(I)$  is embedded continuously in  $L^\infty(I)$ . However, in the higher dimensional case we saw examples, for which this type of embedding is no longer true. To address this issue we start with the situation where  $\Omega = \mathbb{R}^N$ .

**Lemma.** Suppose, for  $N \geq 2$ , that  $f_1, f_2, \dots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$ . Then

$$f(x) = f_1(x_2, \dots, x_N) f_2(x_1, x_3, \dots, x_N) \dots f_N(x_1, x_2, \dots, x_{N-1}) \in L^1(\mathbb{R}^N)$$

and

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \prod_{i=1}^N \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}$$

**Proof.** The case  $N = 2$  is trivial. Consider the case  $N = 3$

$$\begin{aligned} \int |f(x)| dx_3 &= |f_3(x_1, x_2)| \int |f_1(x_2, x_3)| |f_2(x_1, x_3)| dx_3 \\ &\leq |f_3(x_1, x_2)| \left( \int |f_1(x_2, x_3)|^2 dx_3 \right)^{\frac{1}{2}} \left( \int |f_2(x_1, x_3)|^2 dx_3 \right)^{\frac{1}{2}}. \end{aligned}$$

We then integrate with respect to  $x_1$

$$\begin{aligned} \int \int |f(x)| dx_3 dx_1 &\leq \int |f_3(x_1, x_2)| \left( \int |f_1|^2 dx_3 \right)^{\frac{1}{2}} \left( \int |f_2|^2 dx_3 \right)^{\frac{1}{2}} dx_1 \\ &\leq \left( \int |f_1|^2 dx_3 \right)^{\frac{1}{2}} \int |f_3| \left( \int |f_2|^2 dx_3 \right)^{\frac{1}{2}} dx_1 \end{aligned}$$

By using Cauchy-Schwarz, we get

$$\int \int |f| dx_3 dx_1 \leq \left( \int |f_1|^2 dx_3 \right)^{\frac{1}{2}} \left( \int |f_3|^2 dx_1 \right)^{\frac{1}{2}} \|f_2\|_{L^2(\mathbb{R}^2)}.$$

Now, we integrate with respect to  $x_2$ , so we obtain

$$\begin{aligned} \int \int \int |f| dx &\leq \|f_2\|_{L^2(\mathbb{R}^2)} \int \left( \int |f_1|^2 dx_3 \right)^{\frac{1}{2}} \left( \int |f_3|^2 dx_1 \right)^{\frac{1}{2}} dx_2 \\ &\leq \|f_2\|_{L^2(\mathbb{R}^2)} \|f_1\|_{L^2(\mathbb{R}^2)} \|f_3\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

For  $N > 3$ , we use induction. Assume that the assertion of the lemma is true for  $N - 1$  and prove it for  $N$ .

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} |f| dx_1 \dots dx_{N-1} &= \int_{\mathbb{R}^{N-1}} |f_1| \dots |f_{N-1}| |f_N| dx_1, \dots, dx_{N-1} \\ &\leq \left( \int_{\mathbb{R}^{N-1}} |f_N|^{N-1} dx_1 \dots dx_{N-1} \right)^{\frac{1}{N-1}} \left( \int_{\mathbb{R}^{N-1}} |f_1|^{N^1} \dots |f_{N-1}|^{N^1} dx_1 \dots dx_{N-1} \right)^{\frac{1}{N^1}} \end{aligned}$$

by Holder's inequality, where

$$\frac{1}{N-1} + \frac{1}{N^1} = 1 \Rightarrow N^1 = \frac{N-1}{N-2}.$$

Since  $f_i \in L^{N-1}(\mathbb{R}^{N-1}) \Rightarrow$

$$|f_i|^{\frac{N-1}{N-2}} \in L^{N-2}(\mathbb{R}^{N-2}), \text{ for each fixed } , x_N.$$

We then apply the induction hypothesis to get

$$\int_{\mathbb{R}^{N-1}} |f_1|^{\frac{N-1}{N-2}} \dots |f_{N-1}|^{\frac{N-1}{N-2}} dx_1, \dots dx_{N-1} \leq \prod_{i=1}^{N-1} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-2})}^{\frac{N-1}{N-2}}$$

Hence

$$\int_{\mathbb{R}^{N-1}} |f| dx_1 \dots dx_N \leq \|f_N\|_{L^{N-1}(\mathbb{R}^{N-1})} \prod_{i=1}^{N-1} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-2})}.$$

The function

$$F_i(x_N) = \|f_i\|_{L^{N-1}(\mathbb{R}^{N-2})}, \quad 1 \leq i \leq N-1,$$

belongs to  $L^{N-1}(\mathbb{R})$  since

$$\int_{\mathbb{R}} |F_i(x_N)|^{N-1} dx_N = \int_{\mathbb{R}} \int_{\mathbb{R}^{N-2}} |f_i|^{N-1} dx_1 \dots dx_N < \infty$$

by hypothesis. Therefore, as a consequence of Holder's inequality, we have

$$\prod_{i=1}^{N-1} F_i \in L^1(\mathbb{R})$$

which gives, by integration,

$$\int_{\mathbb{R}} \prod_{i=1}^{N-1} F_i dx_N \leq \prod_{i=1}^{N-1} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}$$

hence

$$\int_{\mathbb{R}^N} |f| dx \leq \prod_{i=1}^N \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}.$$

**Theorem** (Sobolev, Gagliardo, Nirenberg)

Suppose that  $1 \leq p < N$ . Then

$$W^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N), \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

Moreover there exists a constant  $C = C(N, p)$  such that

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p}, \quad \forall u \in W^{1,p}(\mathbb{R}^N)$$

**Proof.** Let  $v \in C_0^1(\mathbb{R}^N)$ , so we have

$$|v(x_1, x_2, \dots, x_N)| = \left| \int_{-\infty}^t \frac{\partial v}{\partial x_1}(t, x_2, \dots, x_N) dt \right| \leq \int_{-\infty}^{\infty} \left| \frac{\partial v}{\partial x_1}(t, x_2, \dots, x_N) \right| dt$$

Similarly, we have for  $1 \leq i \leq N$

$$|v(x)| \leq \int_{-\infty}^{\infty} \left| \frac{\partial v}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_N) \right| dt = f_i(\tilde{x}_i),$$

where  $\tilde{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$ . Thus

$$|v(x)|^N \leq \prod_{i=1}^N f_i(\tilde{x}_i),$$

or

$$|v(x)|^{\frac{N}{N-1}} \leq \prod_{i=1}^N f_i^{\frac{1}{N-1}}(\tilde{x}_i).$$

Since each  $g_i = f_i^{\frac{1}{N-1}}(x_i) \in L^{N-1}(\mathbb{R}^{N-1})$ , then  $|v(x)|^{\frac{N}{N-1}} \in L^1(\mathbb{R}^N)$  by the previous lemma and

$$\int_{\mathbb{R}^N} |v(x)|^{\frac{N}{N-1}} \leq \prod_{i=1}^N \|f_i\|_{L^1(\mathbb{R}^{N-1})}^{\frac{1}{N-1}} = \prod_{i=1}^N \left\| \frac{\partial v}{\partial x_i} \right\|_{L^1(\mathbb{R}^N)}^{\frac{1}{N-1}}$$

hence

$$\|v\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \leq \prod_{i=1}^N \left\| \frac{\partial v}{\partial x_i} \right\|_{L^1(\mathbb{R}^N)}^{\frac{1}{N}}$$

We then take  $v = u^{r-1}u$ , for  $r \geq 1$ , we have

$$\begin{aligned} \|u\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)}^r &\leq r \prod_{i=1}^N \left\| u^{r-1} \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^N)}^{\frac{1}{N}} \\ &\leq r \|u\|_{L^{p'(r-1)}}^{r-1} \prod_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p}^{\frac{1}{N}} \end{aligned}$$

We choose then  $r$  in such a way that

$$\frac{rN}{N-1} = p'(r-1);$$

which gives

$$r = \frac{N-1}{N} p^*, \quad p^* = \frac{Np}{N-p}.$$

Consequently we have

$$\|u\|_{L^{p^*}} \leq r \prod_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p}^{\frac{1}{N}} = \frac{p(N-1)}{N-p} \|\nabla u\|_{L^p} \quad \forall u \in C_0^1(\mathbb{R}^N).$$

Now for  $u \in W^{1,p}(\mathbb{R}^N)$ , there exists a sequence  $(u_n) \subset C_0^1(\mathbb{R}^N)$  such that  $u_n \rightarrow u$  in  $W^{1,p}(\mathbb{R}^N)$  and  $u_n \rightarrow u$  a.e. (taking a subsequence if needed). So

$$\|u_n\|_{L^{p^*}} \leq \frac{p(N-1)}{N-p} \|\nabla u_n\|_{L^p}.$$

By using Fatou's lemma and taking  $n$  to  $\infty$ , we get

$$\|u\|_{L^{p^*}} \leq \frac{p(N-1)}{N-p} \|\nabla u\|_{L^p}.$$

**Corollary:** For  $1 \leq p < N$ , then

$$W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) \quad \forall q \in [p, p^*].$$

**Proof.**  $q = \alpha p + (1-\alpha)p^*$ ,  $0 \leq \alpha \leq 1$ ,

$$|u|^q = |u|^{\alpha p} \cdot |u|^{(1-\alpha)p^*} \Rightarrow \int |u|^q = \int |u|^{\alpha p} \cdot |u|^{(1-\alpha)p^*}$$

We use Holder's inequality to get

$$\begin{aligned} \int |u|^q &\leq \left( \int u^p \right)^\alpha \left( \int |u|^{p^*} \right)^{1-\alpha} \\ &\leq C \|u\|_{L^p}^{\alpha p} \|\nabla u\|_{L^p}^{(1-\alpha)p^*} \leq C \|u\|_{W^{1,p}}^q \end{aligned}$$

Hence

$$\|u\|_{L^q} \leq C \|u\|_{W^{1,p}}.$$

**Corollary:** (Case  $p = N$ )

$$W^{1,N}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \quad \forall q \in [N, +\infty)$$

with continuous embedding.

**Proof.** Suppose that  $u \in C_0^1(\mathbb{R}^N)$ , we then use the inequality

$$\|u\|_{L^{\frac{rN}{(N-1)}}}^r \leq r \|u\|_{L^{\frac{r-1}{(N-1)}N}}^{r-1} \prod_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^N}^{\frac{1}{N}}$$

hence we have

$$\|u\|_{L^{\frac{rN}{(N-1)}}}^r \leq r \|u\|_{L^{\frac{r-1}{(N-1)}N}}^{r-1} \|\nabla u\|_{L^N}, \quad \forall r \geq 1$$

Young's inequality then gives

$$\begin{aligned} \|u\|_{L^{\frac{rN}{(N-1)}}}^r &\leq C_1 \|u\|_{L^{\frac{r-1}{(N-1)}N}}^r + C_2 \|\nabla u\|_{L^N}^r \\ &\leq C \left( \|u\|_{L^{\frac{r-1}{(N-1)}N}} + \|\nabla u\|_{L^N} \right)^r \end{aligned}$$

Therefore

$$\|u\|_{L^{\frac{rN}{(N-1)}}} \leq C \left[ \|u\|_{L^{\frac{r-1}{(N-1)}N}} + \|\nabla u\|_{L^N} \right]$$

By choosing  $r = N$ , we obtain

$$\|u\|_{L^{\frac{N}{2}(N-1)}} \leq C \|u\|_{W^{1,N}}$$

By using the interpolation result, then it comes that

$$\|u\|_{L^q} \leq C \|u\|_{W^{1,N}}, \quad \forall q \in \left[ N, \frac{N^2}{N-1} \right]$$

we then take  $r = N + 1, N + 2, \dots$ , etc. to obtain

$$\|u\|_{L^q} \leq C \|u\|_{W^{1,N}}, \quad \forall q \in [N, +\infty).$$

**Theorem.**(Morrey)

Let  $p > N$ , then  $W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$  with continuous embedding. Moreover, we have

$$|u(x) - u(y)| \leq C |x - y|^\alpha \|\nabla u\|_{L^p}, \quad a.e. \ x, y \in \mathbb{R}^N$$

where

$$\alpha = 1 - \frac{N}{p} \text{ and } C = C(N, p).$$

**Remark.** The above inequality implies the existence of a function  $\tilde{u} \in C^{0,\alpha}(\mathbb{R}^N)$  such that  $u = \tilde{u}$  for almost every  $x, y \in \mathbb{R}^N$ . We then say that  $W^{1,p}$ , for  $p > N$ , functions are Holder continuous.

**Proof.** Let  $u \in C_0^1(\mathbb{R}^N)$  and  $Q_0$  be a cube containing the origin with sides parallel to the axes, with length  $= r$ . So for  $x \in Q$ , we have

$$\begin{aligned} |u(x) - u(0)| &= \left| \int_0^1 \frac{d}{dt}(u(tx)) dt = \int_0^1 \sum_{i=1}^N x_i \frac{\partial u}{\partial x_i}(tx) dt \right| \\ &\leq \int_0^1 \sum_{i=1}^N |x_i| \left| \frac{\partial u}{\partial x_i}(tx) \right| dt \leq r \sum_{i=1}^N \int_0^1 \left| \frac{\partial u}{\partial x_i}(tx) \right| dt. \end{aligned}$$

If

$$\bar{u} = \frac{1}{|Q_0|} \int_{Q_0} u(x) dx, \quad |Q_0| = \int_{Q_0} dx = r^N$$

then

$$\begin{aligned} |\bar{u} - u(0)| &= \frac{1}{|Q_0|} \left| \int_{Q_0} (u(x) - u(0)) dx \right| \leq \frac{1}{|Q_0|} \int_{Q_0} |u(x) - u(0)| dx \\ &\leq \frac{r}{|Q_0|} \int_0^1 \sum_{i=1}^N \int_{Q_0} \left| \frac{\partial u}{\partial x_i}(tx) \right| dx dt \\ &\leq \frac{1}{r^{N-1}} \int_0^1 \sum_{i=1}^N \int_{tQ_0} \left| \frac{\partial u}{\partial x_i}(y) \right| \frac{1}{t^N} dy dt, \quad (tx = y) \end{aligned}$$

We then use Holder's inequality to estimate

$$\begin{aligned} \int_{tQ_0} \left| \frac{\partial u}{\partial x_i}(y) \right| dy &\leq \left( \int_{tQ_0} \left| \frac{\partial u}{\partial x_i} \right|^p \right)^{\frac{1}{p}} \left( \int_{tQ_0} 1 \right)^{\frac{1}{p'}} \\ &\leq \left\| \frac{\partial u}{\partial x_i} \right\|_p (tr)^{\frac{N}{p'}} = \left\| \frac{\partial u}{\partial x_i} \right\|_p (tr)^{\frac{N(p-1)}{p}} \end{aligned}$$

Therefore we get

$$\begin{aligned} |\bar{u} - u(0)| &\leq \frac{1}{r^{N-1}} r^{N(p-1)/p} \|\nabla u\|_{L^p(Q_0)} \int_0^1 \frac{t^{N(p-1)/p}}{t^N} dt \\ &\leq \frac{r^{\frac{N}{p}}}{1 - \frac{N}{p}} \|\nabla u\|_{L^p(Q_0)} \end{aligned}$$

But this last inequality remains valid, by translation, for any cube  $Q$  with sides of length  $r$  and parallel to the axes, hence we obtain for any  $x_0$  in this cube

$$|\bar{u} - u(x_0)| \leq \frac{r^{1-\frac{N}{p}}}{1 - \frac{N}{p}} \|\nabla u\|_{L^p(Q)}$$

Thus, for  $x_0, y_0 \in Q$ , we get

$$|u(x_0) - u(y_0)| \leq 2 \frac{r^{1-\frac{N}{p}}}{1 - \frac{N}{p}} \|\nabla u\|_{L^p(Q)}$$

Now for any  $x, y \in \mathbb{R}^N$ , we can find a cube  $Q$  with sides of length  $r = 2|x - y|$  parallel to the axes and containing  $x, y$ ; consequently

$$\begin{aligned} |u(x) - u(y)| &\leq 2 \frac{2^{1-\frac{N}{p}} |x - y|^{1-\frac{N}{p}}}{1 - \frac{N}{p}} \|\nabla u\|_{L^p(Q)} \\ &\leq C |x - y|^{1-\frac{N}{p}} \|\nabla u\|_{L^p(\mathbb{R})} \end{aligned}$$

for any  $u \in C_0^1(\mathbb{R}^N)$ .

For  $u \in W^{1,p}(\mathbb{R}^N)$ , we use a sequence  $(u_n) \subset C_0^1(\mathbb{R}^N)$  such that  $u_n \rightarrow u$  in  $W^{1,p}(\mathbb{R}^N)$  and  $u_n \rightarrow u$  a.e.  $x$  in  $\mathbb{R}^N$ . Hence the second assertion of the theorem is established.

To establish the  $L^\infty$  bound, we use

$$|u(x) - \bar{u}| \leq C \|\nabla u\|_{L^p(Q)},$$

which implies for a cube containing  $x$  and with  $r = 1$ ,

$$|u(x)| \leq |\bar{u}| + \frac{1}{1 - \frac{N}{p}} \|\nabla u\|_{L^p(Q)}$$

By using

$$|\bar{u}| \leq \frac{1}{|Q|} \int |u(y)| dy \leq \|u\|_{L^p(Q)} |Q| = \|u\|_{L^p(Q)}$$

We arrive at

$$|u(x)| \leq \frac{1}{1 - \frac{N}{p}} \left( \|\nabla u\|_{L^p(Q)} + \|u\|_{L^p(Q)} \right) \leq \frac{1}{1 - \frac{N}{p}} \|u\|_{W^{1,p}(\mathbb{R}^N)}$$

for any  $u \in C_0^\infty(\mathbb{R}^N)$ .

If  $u \in W^{1,p}(\mathbb{R}^N)$ , we then approximate it by a sequence  $(u_n)$  in  $C_0^\infty(\mathbb{R}^N)$  which converges to  $u$  in  $W^{1,p}(\mathbb{R}^N)$  and almost everywhere. Thus we obtain the desired result.

**Corollary.** If  $u \in W^{1,p}(\mathbb{R}^N)$ ,  $N < p < \infty$ . Then  $\lim_{|x| \rightarrow \infty} u(x) = 0$ .

**Proof.** We approximate by a sequence  $(u_n) \subset C_0^\infty(\mathbb{R}^N)$ . So,  $\lim_{n \rightarrow \infty} u_n = u$  in  $W^{1,p}(\mathbb{R}^N)$  and  $\lim_{n \rightarrow \infty} \|u_n - u\|_\infty = 0$ . So

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{R}$  such that  $\forall n > n_0, |u_n(x) - u(x)| < \varepsilon, \forall x \in \mathbb{R}^N$ , which implies that

$$|u(x)| < |u_n(x)| + \varepsilon, \quad \forall x \in \mathbb{R}^N, \quad \forall n \geq n_0.$$

If  $|x| \rightarrow \infty$  then  $u_n(x) \rightarrow 0$ , therefore

$$\lim_{|x| \rightarrow \infty} |u(x)| < \varepsilon, \quad \forall \varepsilon > 0 \Rightarrow \lim_{|x| \rightarrow \infty} |u(x)| = 0.$$