Lemma. (Partition of Unity)

Let $S \subset \mathbb{R}^N$ be compact and $O_1, O_2, \ldots O_k$ be open covering S; that is $S \subset \bigcup_{i=1}^{k} O_i$.
Then there exist functions $\psi_0, \psi_1, \ldots, \psi_k \in C^\infty(\mathbb{R}^N)$ such that

(i) $0 \le \psi_i \le 1, \quad \forall\, i = 0, 1, 2, \ldots, k$ and $\displaystyle\sum_{i=0}^{k} \psi_i = 1.$

(ii) *supp* $\psi_i \subset O_i, \quad \forall\, i = 1, 2, \ldots, k$ and *supp* $\psi_o \subset \mathbb{R}^N \backslash S$

Proof. For each $x \in S$, there exists r_x such that the ball $B(x, 2r_x) \subset O_i$, for some $i \in \{1, 2, \ldots, k\}$.

Since S is compact then there exist a finite number of balls $(B(x_i, r_{x_i}))$ covering S; that is $S \subset \bigcup_{i=1}^{m} B(x_i, r_{x_i})$.

Let $\varepsilon = \min_{1\le i\le m} r_{x_i}$ and set $O_{i,\varepsilon} = \{x \in O_i / \text{ distance } (x, \partial O_i) > \varepsilon\}$, $i = 1, 2, \ldots k$. it is easy to check that $\{O_{i,\varepsilon}\}_{i=1}^{k}$ is a covering of S. Let

$$O_1' = O_{1,\varepsilon}\,, \quad O_j' = O_j \setminus \bigcup_{i=1}^{k-1} O_{i,\varepsilon}, \quad \forall_j = 2, \ldots k \text{ and } O_0 = \bigcup_{i=1}^{k} O_i = \bigcup_{i=1}^{k} O_{i,\varepsilon}$$

Let $\chi_i = \chi_{O_i'}$ (the characteristic function), so $\sum_{i=1}^{k} \chi_i = 1$ on $O_0 \supset S$ since (O_i') are pairwise disjoint. define

$$\psi_i = \rho_h * \chi_i, \qquad h < \min\left\{\frac{\varepsilon}{2}, \text{ dist } (S, \partial\, O_0)\right\}$$

So

$$\psi_i \in C^\infty(\mathbb{R}^N) \text{ and } \psi_i(x) = 0, \quad \forall x / \text{ dist } (x, O_i') > h$$

hence $\psi_i \in C_0^\infty(O_i)$. Also

$$\begin{aligned}\sum_{i=1}^{k} \psi_i &= \sum_{i=1}^{k} \int_{\mathbb{R}^N} \rho_h(x-y)\psi_i(y)dy \\ &= \int_{\mathbb{R}^N} \rho_h(x-y) \sum_{i=1}^{k} \psi_i(y)dy = \int_{\mathbb{R}^N} \rho_h = 1, \quad \forall\, x \in O_o \supset S.\end{aligned}$$

Proposition (Change of variables)

Let $H : \Omega' \longrightarrow \Omega$ be a bijection with Ω, Ω' opens of $\mathbb{R}^N$ and such that

$$H \in C^1(\Omega'), \quad H^{-1} \in C^1(\Omega), \quad \text{Jac}\, H \in L^\infty(\Omega'), \quad \text{Jac}\, H^{-1} \in L^\infty(\Omega)$$

If $u \in W^{1,p}(\Omega)$ then $uoH \in W^{1,p}(\Omega')$ with

$$\frac{\partial}{\partial y_i}(uoH)(y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(H(y)) \frac{\partial H_i}{\partial y_j}(y), \quad 1 \le i \le N.$$

Here $H(y) = x$.
Proof: For $1 \leq p < \infty$, we choose a sequence $(u_n) \in C_0^\infty(\mathbb{R}^N)$ such that $u_n \longrightarrow u$ in $L^p(\Omega)$ and $\nabla u_n \longrightarrow \nabla u$ in $[L^p(\omega)]^N$, $\quad \forall \omega \subset\subset \Omega$.. So

$$u_n oH \longrightarrow uoH \text{ in } L^p(\Omega^{'})$$

and

$$\frac{\partial u_n}{\partial x_i} oH \longrightarrow \frac{\partial u}{\partial x_i} oH \text{ in } L^p(\omega^{'}), \quad \forall \omega' \subset\subset \Omega'$$

By taking $\phi \in C_0^1(\Omega')$, we easily see that

$$\int_{\Omega'} (u_n oH) \frac{\partial \phi}{\partial y_j} = -\int_{\Omega'} \sum_{i=1}^N \left(\frac{\partial u_n}{\partial x_i} oH \right) \frac{\partial H_i}{\partial y_j} \phi$$

By letting n go to ∞, we arrive to the desired result. For $p = +\infty$, we proceed like the previous theorems.
Theorem. (Extension Theorem)
Suppose that Ω is of class C^1 with $\partial\Omega$ bounded $\left(\text{or } \Omega = \mathbb{R}_+^N\right)$. Then there exists an extension operator

$$P : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^N)$$

linear and such that $\forall\, u \in W^{1,p}(\Omega)$

(i) $Pu_{|\Omega} = u$

(ii) $||Pu||_{L^p(\mathbb{R}^N)} \leq C||u||_{L^p(\Omega)}$

(iii) $||Pu||_{W^{1,p}(\mathbb{R}^N)} \leq C||u||_{W^{1,p}(\Omega)}$.

C is a constant depending on p and Ω.
Proof. Since $\partial\Omega$ is compact and of class C^1, then there exists k opens $(O_i)_{i=1}^k$ such that $\partial\Omega \subset \bigcup_{i=1}^k O_i$ and bijections $H_i : Q \longrightarrow O_i$ such that

$$H_i \in C^1(\bar{Q}), \quad H_i^{-1} \in C^1\left(\bar{O}_i\right), \quad H_i(Q_+) = O_i \cap \Omega$$

and

$$H_i(Q_0) = O_i \cap \partial\,\Omega$$

Consider the functions $\theta_0, \theta_1, \ldots, \theta_k$ seen in the partition of unity lemma. We then set

$$u = u \sum_{i=0}^k \theta_i = \sum_{i=0}^k u\theta_i = \sum_{i=0}^k u_i, \qquad u_i = u\theta_i$$

Extension of $\mathbf{u_0}$: Let

$$\tilde{u}_0(x) = \begin{cases} u_0(x) & , \quad x \in \Omega \\ 0 & , \quad x \in \mathbb{R}^N \backslash \Omega \end{cases}$$

and note that

$$\theta_0 \in C^1(\mathrm{I\!R}^N) \cap L^\infty(\mathrm{I\!R}^N), \quad 0 \leq \theta_0 \leq 1, \quad \nabla\theta_0 \in \left[L^\infty(\mathrm{I\!R}^N)\right]^N$$

since

$$\nabla\theta_0 = -\sum_{i=1}^{k} \nabla\theta_i, \qquad supp\,\theta_i \subset O_i, \quad \forall\, i = 1, 2, \ldots k,$$

and $supp\,\theta_i$ is compact. Therefore

$$\tilde{u}_0 \in W^{1,p}(\mathrm{I\!R}^N), \qquad \frac{\partial}{\partial x_i}\,\tilde{u}_0 = \theta_0 \widetilde{\frac{\partial u}{\partial x_i}} + \tilde{u}\,\frac{\partial \theta_0}{\partial x_i},$$

hence

$$||u_0||_{W^{1,p}(\mathrm{I\!R}^N)} \leq C||u||_{W^{1,p}(\Omega)},$$

where C is constant depending on the L^∞ norm of θ_0 and $\nabla\theta_0$.

Extension of $\mathbf{u}_i$, $\mathbf{1 \leq i \leq k}$:

We consider the restriction of u to $O_i \cap \Omega$ and we "transport" it over Q_+ by using H_i. For this, we define

$$v_i(y) = u(H_i(y)), \quad \forall y \in Q_+.$$

It is easy to verify that $v_i \in W^{1,p}(Q_+)$. We then extend v_i to Q by reflection and denote this extension by v_i^*, which belongs to $W^{1,p}(Q)$. We then "retransport back" v_i^* over O_i by using H_i^{-1}. Let

$$w_i(x) = v_i^*\left[H_i^{-1}(x)\right], \quad \forall x \in O_i;$$

So

$$w_i \in W^{1,p}(O_i) \text{ and } w_i = u \text{ over } O_i \cap \Omega$$

with

$$||w_i||_{W^{1,p}(O_i)} \leq C||u||_{w^{1,p}(O_i \cap \Omega)}$$

Finally let

$$\hat{u}_i(x) = \begin{cases} \theta_i(x) w_i(x) & , \quad \forall x \in O_i \\ 0 & , \quad \forall x \in \mathrm{I\!R}^N \backslash O_i \end{cases}$$

By the above lemma, we have

$$\hat{u}_i \in W^{1,p}(\mathrm{I\!R}^N), \quad \hat{u}_i = u_i \text{ over } \Omega$$

and

$$||\hat{u}_i||_{W^{1,p}(\mathrm{I\!R}^N)} \leq C||u||_{W^{1,p}(O_i \cap \Omega)}$$

Conclusion. $Pu = \tilde{u}_0 + \sum_{i=1}^{k} \hat{u}_i$ has all desired properties.

Corollary (Density):

Suppose that Ω is of class C^1 and let $u \in W^{1,p}(\Omega)$ be given with $1 \leq p < +\infty$. Then there exists a sequence $(u_n) \subset C_0^\infty(\mathbb{R}^N)$ such that

$$u_{n|\Omega} \longrightarrow u \text{ in } W^{1,p}(\Omega).$$

Proof.

(1) If $\partial\Omega$ is bounded we then extend u to $\mathbb{R}^N$ and then we take $u_n = \xi_n (\rho_n * Pu)$, which converges to Pu in $W^{1,p}(\mathbb{R}^N)$. In particular $u_{n|\Omega} \longrightarrow u$ in $W^{1,p}(\Omega)$.

(2) If $\partial\Omega$ is unbounded, then we consider the sequence $\xi_n u$, where $\xi_n = \xi\left(\frac{x}{n}\right)$ and ξ is the truncation function. We know that $\xi_n u \longrightarrow u$ in $W^{1,p}(\Omega)$, so for $\varepsilon > 0, \quad \exists n_0 \in \mathbb{N}$ such that $||\xi_{n_0} u - u||_{W_{1,p}(\Omega)} < \frac{\varepsilon}{2}$.

Since *supp* $\xi_{n_0} u$ is included in a large ball we then extend $\xi_{n_0} u$ to $\mathbb{R}^N$ and apply the above to get a function v_ε in $C_0^\infty(\mathbb{R}^N)$ such that $||v_\varepsilon - \xi_{n_0} u||_{W^{1,p}} < \frac{\varepsilon}{2}$. Consequently

$$||u_0 - v_\varepsilon||_{W^{1,p}(\Omega)} < \varepsilon$$

We then construct the sequence $(v_\varepsilon) \subset C_0^\infty(\mathbb{R}^N)$ which converges to u in $W^{1,p}(\Omega)$.