Chapter 2

Sobolev Spaces

2.1 Motivation

Let us consider the problem

$$\left.\begin{array}{ll} -u''(x) + u(x) = f(x), & 0 < x < 1 \\ u(0) = u(1) = 0. & \end{array}\right\} \tag{1}$$

The objective is to find a solution $u \in C^2([0,1])$ such that $f \in C([0,1])$.

Such a problem is solvable by standard calculus methods.

Suppose that $\varphi \in C_9((0,1))$; that φ is continuous and $\varphi(0) = \varphi(1) = 0$. So

$$\int_0^1 (-u'' + u)\varphi(x)dx = \int_0^1 f\varphi dx.$$

Using integration by parts, we obtain

$$\int_0^1 u'\varphi' + u\varphi = \int_0^1 f\varphi dx. \tag{2}$$

We notice here that (2) is valid if u and $u' \in L^1((0,1))$ or "simply" $u \in C^1([0,1])$. In this case, we say that u is a weak solution of (1). It satisfies (1) in the weak or variational sense (2).

(2) is the variational equation for (1).

Theorem 2.1.1 *If $f \in C^0$, then any weak solution $u \in C^1$ is C^2.*

Proof. It suffices to note that $u'' = u - f \in C$. So $u \in C^2$. ■

Theorem 2.1.2 *Any $u \in C^1((0,1))$ with $u(0) = u(1) = 0$ satisfying (2) for all $\varphi \in C_0^1((0,1))$. Then u is a classical solution; that u satisfies (1).*

Proof.

$$\int_0^1 u'\varphi' + u\varphi = \int_0^1 f\varphi$$

by previous theorem $u \in C^2$, so we integrate by parts $\int_0^1 u'\varphi' = -\int u''\varphi + u'\varphi(1) - u'\varphi(0) = -\int_0^1 u''\varphi$. Therefore, we have

$$\int_0^1 (-u'' + u - f)\varphi(x)dx = 0$$

for all $\varphi \in C_0^1((0,1))$ which implies that $-u'' + u - f = 0$ for a.e. and since u, u'', f are continuous $\Rightarrow$

$$-u'' + u = f, \quad \forall\, x \in (0,1).$$

Hence (1) is satisfied. ■

Remark 2.1.1 In some instance, we can show that a weak solution is in fact a classical solution. This is called the regularity theory. It will be a part of our course.

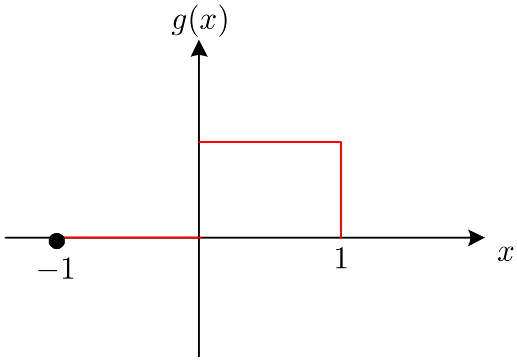
2.2 Sobolev Space $W^{1,p}(I)$

Definition 2.2.1 Given an open interval I and $1 \leq p \leq +\infty$ (bounded or not). We define the Sobolev space $W^{1,p}(I) = \{u \in L^p(I)$ such that there exists $g \in L^p(I)$ for which we have $\int_I u\varphi' = -\int_I g\varphi, \quad \forall\, \varphi \in C_0^1(I)\}$.

Remark 2.2.1 For $p = 2$, we denote $W^{1,2}_{(I)} = H^1(I)$.

Remark 2.2.2 In such a case, we write $u' = g$ in the weak sense; that is

$$\int_I (u' - g)\varphi dx = 0, \quad \forall\, \varphi C_0^1(I).$$



Example 2.2.1 Let $I = (-1,1)$ and $u(x) = \dfrac{1}{2}(x + |x|)$. It is clear that u is not differentiable at $x = 0$ (hence on I).

Let $\varphi \in C_0^1(I)$, then

$$\begin{aligned}\int_{-1}^{1} u\varphi' &= \int_{-1}^{0} \varphi'(0)dx + \int_0^1 x\varphi' dx = \int_0^1 x\varphi' dx \\ &= x\varphi|_0^1 - \int_0^1 \varphi(x)dx = \int_{-1}^{1} g\varphi(x)dx\end{aligned}$$

where

$$g(x) = \begin{cases} 0, & -1 \leq x < 0 \\ 1, & 0 \leq x < 1. \end{cases}$$

This Heavyside function denoted by H. So $u' = H$ is the weak sense.

Note that $u, H \in L^p(I), \quad \forall\, p \geq 1 \Rightarrow u \in W^{1,p}(I), \quad \forall\, 1 \leq p \leq \infty.$

Now let us consider $H(x)$ and test if it is $W^{1,p}(I)$ function. For this take $\varphi \in C_0^1(I)$ a test function

$$\int_{-1}^{1} H\varphi' dx = \int_0^1 \varphi'(x) = \varphi(1) - \varphi(0) = -\varphi(0) = -\delta(\varphi).$$

In this case, we cannot find $g \in L^p(I)$ such that

$$\int_{-1}^{1} g\varphi dx = -\varphi(0), \quad \forall\, \varphi \in C_0^1(I).$$

Remark 2.2.3 In this latter case, we say that $H^1 = \delta$ in the distributional sense. δ is a distribution. This also motivates us to define $W^{1,p}(I)$ by using distributions.

Definition 2.2.2 We say that $u \in W^{1,p}(I)$ if $u \in L^p(I)$ and its distribution derivative u' coincides with an $L^p(I)$ function g.

Theorem 2.2.1 *The Sobolev space $W^{1,p}(I)$ equipped with the norm $\|W\|_{1,p} = \|u\|_p + \|u'\|_p$ is a Banach space.*

Proof. Let (u_n) be a Cauchy sequence in $W^{1,p}(I)$. So (u_n) and (u'_n) are Cauchy in $L^p \Rightarrow$

$$u_n \to u \text{ and } u'_n \to g \text{ in } L^p(I).$$

Now

$$\int_I u_n \varphi' = -\int_I u'_n \varphi, \quad \forall\ n\varphi \in C_0^1(I).$$

As $n \to \infty$,

$$\int_I u\varphi' = -\int_I g\varphi, \quad \forall\ \varphi \in C_0^1(I)$$

$\Rightarrow u \in W^{1,p}(I)$ with $u' = g$ (Definition) and $\|u_n - u\|_{1,p} \to 0$ as $n \to \infty$. ■

Theorem 2.2.2 *The space $W^{1,p}(I)$ is reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$.*

Proof. We define the operator

$$T : W^{1,p}(I) \to L^p(I) \times L^p(I)$$

$$u \to (u, u').$$

This is an isometry. So $T(W^{1,p}(I))$ is a closed subspace of $L^p(I) \times L^p(I)$; hence $T(W^{1,p}(I))$ is reflexive for $1 < p < \infty$ since $L^p \times L^p$ is reflexive for $1 < p < \infty$.

The same thing holds for separability. ■

Theorem 2.2.3 *Given $u \in W^{1,p}(I)$. There exists $\tilde{u} \in C(\overline{I})$ such that $\tilde{u} = u$ a.e. in I and $\tilde{u}(g) - \tilde{u}(x) = \int_x^y u'(t)dt \quad \forall\ x, y \in I$.*

Proof. It is clear that $\tilde{u}(x) = u(x_0) + \int_{x_0}^{x} u'(t)dt, \quad x_0 \in I$, is absolutely continuous $\Rightarrow \tilde{u} \in C(\overline{I})$. Moreover, $\tilde{u}' = u'$ a.e. in $I \Rightarrow \tilde{u} = u$ a.e. in I. ■

Remark 2.2.4 This theorem shows that $W^{1,p}(I)$ functions can be represented by absolutely continuous functions defined on $\overline{I}$. This is why we usually say that if $u \in W^{1,p}(I)$ then $u \in C(\overline{I})$.

Remark 2.2.5 If u is absolutely continuous with $u' \in L^p(I)$, this does not mean that $u \in W^{1,p}(I)$ unless I is bounded.

Let $g(t) = \dfrac{1}{1+t^2} \in L^p(\mathbb{R}^+), \quad \forall\ p \geq 1$, but $u(x) = \int_0^x g(t)dt = \tan^{-1} x$ is not in $L^p(\mathbb{R}^+)$. Since $\int_0^R \tan^{-1} x dx = R\left[\tan^{-1} R - \dfrac{\log(1+R^2)}{2R}\right]$. As $R \to \infty$, $\int_0^\infty \tan^{-1} x dx = \infty$.

Definition 2.2.3 Let u be defined in I; for x and h such that $x + h \in I$, we denote by $\tau_h f(x) = f(x+h)$.

Theorem 2.2.4 *Let $u \in L^p(I), \quad 1 < p \leq \infty$. The following properties are equivalent:*

(i) $u \in W^{1,p}(I)$.

(ii) There exists a constant $C > 0$ such that

$$\left|\int_I u\varphi'\right| \leq C\|\varphi\|_{p'}, \quad \forall\ \varphi \in C_0^\infty(I), \quad \frac{1}{p'} + \frac{1}{p} = 1.$$

(iii) There exists a constant $C' > 0$ such that for each open $\omega \subset\subset I(\overline{\omega} \subset I)$ and for each h with $|h| <$ distance $(\omega, \partial I)$, we have

$$\{\tau_h u - u\|_p \leq C'|h|.$$

Proof. (i) $\Rightarrow$ (ii) by Holder's inequality with $c = \|u\|_p$.

(ii) $\Rightarrow$ (i) why?

We define $F : C_0^\infty(I)\mathbb{R}$ by $F(\varphi) = \displaystyle\int_I u\varphi'$, which is continuous linear form.

By Hahn Banach theorem this form is extended to $L^{p'}(I)$. So $|F(\varphi)| \leq C\|\varphi\|_{L^{p'}}, \quad \forall\, \varphi \in L^{p'}$.

By Riesz representation theorem, there exists $g \in L^p(I)$ such that

$$\int_I u\varphi' = \int_I g\varphi, \quad \forall\, \varphi \in L^{p'}(I);$$

in particular, we have $\displaystyle\int_I u\varphi' = \int_I g\varphi, \quad \forall\, \varphi \in C_0^\infty(I)$. So, $u \in W^{1,p}(I)$ with

$u' = -g$

(i) $\Rightarrow$ (iii) why?

$$(x+h) - ux() = \int_x^{x+h} u'(t)dt.$$

Now for

$$p = +\infty \Rightarrow |u(x+h) - u(x)|_\infty \leq \left| \int_x^{x+h} |u'|_\infty dt \right| \; \|\tau_n u\|_\infty \leq \|u\|_\infty |h|.$$

Next $1 < p < \infty$. We write

$$\begin{aligned}
u(x+h) - u)x) &= \int_0^1 hu'(x+fh)ds \Rightarrow \\
|u(x+h) - u(x)|^p &\leq |h|^p \int_0^1 |u'(x+sh)|^p ds \\
\Rightarrow \int_\omega |u(x+h) - u(x)|^p dx &\leq |h|^p \int_\omega |u'(x+sh)|^p ds dx \\
\|\tau_h u - u\|_{L^p} &\leq |h|^p \int_0^1 |u'(x+sh)|^p dx ds \\
&\leq |h|^p \int_1^0 \|u'\|_p^p ds = |h|^p \; \|u'\|_p^p.
\end{aligned}$$

Therefore $\|\tau_h u - u\|_p \leq |h| \; \|u\|_{1,p}$.

(iii) $\Rightarrow$ why?

Let $\varphi \in C_0^\infty(I) \Rightarrow$ supp $\varphi \subset \omega \subset\subset I$. Suppose h such that $|h| <$ distance $(\omega, \partial I)$, so

$$\begin{aligned} \int_I [u(x+h) - u(x)]\varphi(x)dx &= \int_I u(x)[\varphi(x-h) - \varphi(x)]dx \\ \Rightarrow \left| \int_I u(x)[\varphi(x-h) - \varphi(x)]dx \right| &\leq \|\tau_h\|_p \, \|\varphi\|_{p'} \\ &\leq c|h| \, \|\varphi\|_{p'}. \end{aligned}$$

Divide by h and let $h \to 0$

$$\left\| \int_I u(x)\varphi'(x)dx \right\| \leq C\|\varphi\|_{p'}.$$

This completes the proof. ■

Remark 2.2.6 For $p = 1$, we only have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

(ii) $\Rightarrow$ (i) is in general false because of the nonseparability of $L^\infty(I)$ space.

2.3 Extension and Density

Definition 2.3.1 Given a function $f : I \to \mathbb{R}$ and J is an interval such that $I \subset J$. We say that $\tilde{f}$ is an extension of f on J if $\tilde{f}(x) = f(x), \quad \forall\, x \in I$.

Theorem 2.3.1 (Extension Operator)*: Given* $1 \leq p \leq \infty$*. There exists an operator* $P : W^{1,p} \to W^{1,p}(\mathbb{R})$ *such that*

(i) $P_{u_{|_I}} = u, \quad \forall\, u \in W^{1,p}(I)$,

(ii) $\|P_u\|_p \leq C\|u\|_{1,p}, \quad \forall\, u \in W^{1,p}(I)$,

(iii) $\|P_u\|_{1,p} \leq C'\|u\|_{1,p}, \quad \forall\, u \in W^{1,p}(I)$.

Proof.

1. First suppose that $I = (0, +\infty)$. Then take $Pu = u^*$ defined by reflexion as

$$\begin{aligned} P_u(x) &= u^*(x) = \begin{cases} u(x), & \forall\, x \in (0, +\infty) \\ u(-x), & \forall\, x \in (-\infty, 0] \end{cases} \\ \|P_u\|^p_{L^p(\mathbb{R})} &= 2\|u\|^p_{L^p(I)} \Rightarrow \\ \|P_u\|_{L^p(\mathbb{R})} &\leq 2\|u\|_{L^p(I)}. \end{aligned}$$

Also,

$$(P_u)'(x) = \begin{cases} u'(x), & \text{a.e. } x \in (0, +\infty) \\ -u'(-x), & \text{a.e } x \in (-\infty, 0) \end{cases}$$

with $\|(P_u)'\|_{L^p(\mathbb{R})} \leq 2\|u\|_{L^p(I)}$.

2. Suppose that $I = (a, b)$ bounded. We can always take $I = (0, 1)$ by a change of variable ($t = a + s(b - a)$). Let η be a $C^1(\mathbb{R})$ function such that

$$\eta(x) = \begin{cases} 1, & x < 1/4 \\ 0, & x > 3/4 \end{cases}$$

Now we extend u to $(0, +\infty)$ by $\eta\tilde{u}$, where

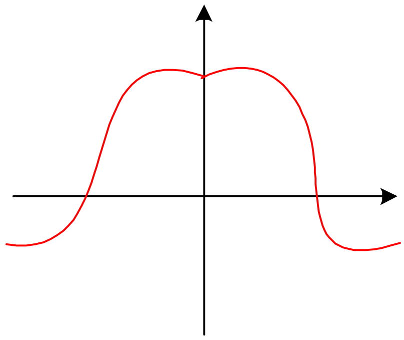
$$\tilde{u}(x) = \begin{cases} u(x), & x \in (0, 1) \\ 0, & x \in [1, +\infty). \end{cases}$$

Let us show that $\eta\tilde{u} \in W^{1,p}((0, +\infty))$

$$\int_0^{+\infty} |\eta\tilde{u}|^p = \int_0^1 |\eta u|^p \leq \int_0^1 |u|^p \Rightarrow \|\eta\tilde{u}\|_{L^p(\mathbb{R}^+)} \leq \|u\|_{L^p(\mathbb{R})}.$$

Let $\varphi \in C_0^\infty(\mathbb{R})$, then

$$\begin{aligned} \int_0^{+\infty} \eta\tilde{u}\varphi'(x)dx &= \int_0^1 \eta u\varphi' dx \\ &= +\int_0^1 u[(\eta\varphi)' - \eta'\varphi]dx \\ &= +\int_0^1 u(\eta\varphi)' - \int_0^1 u\eta'\varphi \\ &= -\int_0^1 u'\eta\varphi - \int u\eta'\varphi = -\int_0^1 (u'\eta + u\eta')\varphi. \end{aligned}$$



since $\eta\varphi \in C_0^1(I)$.

Thus

$$\int_0^{+\infty} \eta\tilde{u}\varphi' = -\int_0^{+\infty} (\tilde{u}'\eta + \tilde{u}\eta')\varphi$$

where

$$\tilde{u}' = \begin{cases} u' & \text{on } I \\ 0 & \text{on } [1, +\infty) \end{cases}$$

and it is easy to see that

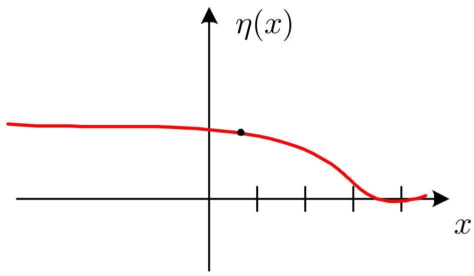
$$\begin{aligned} \|\eta\tilde{u}' + \eta'\tilde{u}\|_{L^p(\mathbb{R}^+)} &= \|\eta u' + \eta' u\|_{L^p(I)} \\ &\leq \|\eta u'\|_{L^p(I)} + \|\eta' u\|_{L^p(I)} \\ &\leq \|u'\|_{L^p(I)} + c\|u\|_{L^p(I)} \end{aligned}$$

$C = \|\eta'\|_\infty$; so

$$\|\eta\tilde{u}\|_{1,p} \leq C\|u\|_{1,p}.$$

Now we are in the first case and hence $\eta\tilde{u}$ is extended to $\mathbb{R}$ by reflexion. We denote by v_1 extension of ηu to $\mathbb{R}$.

3. Similarly, we do the same for $(1-\eta)u$ and we denote by v_2 the extension of $(1-\eta)u$ to $\mathbb{R}$. Let $v = v_1 + v_2$; so $v \in W^{1,p}(\mathbb{R})$ and $v(x) = u(x)$, $\forall\, x \in I$. This completes the proof.



■

Remark 2.3.1 If $u \in W^{1,p}(I),\ I = (a,b)$ extending u by zero outside I, will not give in general $\tilde{u}$ in $W^{1,p}(\mathbb{R})$.

Example 2.3.1 Let $u(x) = x$ on $(,01)$. So

$$\tilde{u}(x) = \begin{cases} x, & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$$

Let $\varphi \in C_0^\infty(\mathbb{R})$

$$\begin{aligned} \int_{-\infty}^{\infty} \tilde{u}\varphi'(x)dx &= \int_0^1 x\varphi'(x)dx = x\varphi(x)|_0^1 - \int_0^1 \varphi(x)dx \\ &= \varphi(1) - \int_0^1 \varphi(x)dx = \delta_1\varphi - \int_0^1 \varphi \\ &= \delta_1\varphi - \int_{-\infty}^{\infty} \tilde{u}'\varphi. \end{aligned}$$

Hence $\tilde{u} \notin W^{1,p}(\mathbb{R})$.

Remark 2.3.2 The extension operator is not unique. It depends heavily on the choice of η.

Definition 2.3.2 (Convolution). Let ρ be in $L^1(\mathbb{R})$ and W in $L^p(\mathbb{R})$. We define the convolution of ρ and W by

$$\rho * g(x) = \int_{\mathbb{R}} \rho(x-y)g(y)dy, \quad \text{for almost every } x \in \mathbb{R}$$

Properties:

1. $\rho * g = g * \rho$.

2. If $\rho \in L^1$ and $g \in L^p$, then $\rho * g$ is in $L^p(\mathbb{R})$. Moreover

$$\|\rho * g\|_{L^p(\mathbb{R})} \leq \|\rho\|_{L^1(\mathbb{R})} \cdot \|g\|_{L^p(\mathbb{R})}.$$

3. If $\rho \in L^1(\mathbb{R}),\ g \in C_0(\mathbb{R})$, then $\rho * g \in L^\infty(\mathbb{R})$ with

$$\tau_h(\rho * g) - \rho * g = \rho * (\tau_h g - g)$$

and

$$\|\rho * (\tau_n g - g)\|_{L^\infty} \leq \|\rho\|_{L^1} \|\tau_h g - g\|_{L^\infty}.$$

Hence

$$\|\rho * (\tau_h g - g)\|_{L^\infty} \to 0 \text{ as } h \to 0.$$

4. If $f \in L^1(\mathbb{R})$ and $\varphi \in C_0^1(\mathbb{R})$ or $C_0^\infty(\mathbb{R})$, then $f*\varphi \in C^1$ or C^∞ with $(f*\varphi)' = f*\varphi'$.

5. If f and φ are of compact support, then $f * \varphi$ is of compact support.

Proof.

1. $\displaystyle\int_{-\infty}^{\infty} \rho(x-y)g(y)dy = -\int_{\infty}^{-\infty} \rho(s)g(x-s)ds \qquad s = x - y \Rightarrow dy = -ds.$

2. **Jension Inequality**.

Lemma: If $f \geq 0$ and $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$ and $\phi \geq 0$ continuous and convex, then

$$\phi\left(\int fg\right) \leq \int f\phi(g),$$

so let $f = \dfrac{|\rho|}{\|\rho\|_{L^1}}$ and $\phi(r) = r^p, \quad p \geq 1$

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{\rho(y)}{\|\rho\|_{L^1}} g(x-y)dy \right|^p dx \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\rho(y)|}{\|\rho\|_{L^1}} |g(x-y)|^p dydx$$
$$\frac{1}{\|\rho\|_{L_1}^p} \|g * \rho\|_{L^1}^p \leq \int_{-\infty}^{\infty} \frac{|\rho(y)|dy}{\|\rho\|_{L^1}} \int_{-\infty}^{\infty} |g(x-y)|^p dx = \|g\|_{L^p}^p$$

3.

$$
\begin{aligned}
|\rho * g(x)| &\leq \int_{\mathbb{R}} \rho(x-y)|\,|g(y)|dy \\
&= \|g\|_{L^\infty} \cdot |\rho\|_{L^1} \Rightarrow
\end{aligned}
$$

$\rho * g \in L^\infty$.

$$
\begin{aligned}
&\tau_h(\rho * g)(x) - g * p(x) \\
&\quad= \int_{\mathbb{R}} \rho(y)g(x+h-y)dy - \int_{\mathbb{R}} \rho(y)g(x-y)dy \\
&\quad= \int_{\mathbb{R}} \rho(y)[g(x+h-y) - g(x-y)]dy \\
&\quad= \int_{\mathbb{R}} \rho(y)[\tau_h g(x-y) - g(x-y)]dy \\
&\quad= \rho * (\tau_h g - g)(x).
\end{aligned}
$$

It is also clear that $\|\rho*(\tau_h g - g)\|_{L^\infty} \leq \|\rho\|_{L^1}\|\tau_h g - g\|_{L^\infty}$ and hence g is continuous, then $\|\tau_h g - g\|_{L_\infty} \to 0$ as $h \to 0$.

4.

$$
\begin{aligned}
&\frac{(f*\varphi)(x+h) - (f*\varphi)(x)}{h} - (f*\varphi')(x) \\
&\quad= \int_{\mathbb{R}} f(y)\left[\frac{\varphi(x+h-y) - \varphi(x-y)}{h} - \varphi'(x-y)\right] dy.
\end{aligned}
$$

and since $\varphi \in C_0^1(\mathbb{R})$, then we take limit to get

$$
\lim_{h\to 0} \frac{(f*\varphi)(x+h) - (f*\varphi)(x)}{h} - (f*\varphi')(x) = 0
$$

hence $(f*\varphi)'(x) = f*\varphi'(x)$.

Now, similarly if $\varphi \in C_0^2(\mathbb{R})$, then

$$
(f*\varphi')' = f*\varphi' \text{ but } (f*\varphi')' = (f*\varphi)''
$$

and we continue to find that

$$
(f*\varphi)^{(m)} = f*\varphi^{(m)}, \quad \forall\; m = 1, 2, 3, \ldots.
$$

5. If supp $f \subset (-r,r)$ and support $\varphi \subset (R,R)$, then supp $f * \varphi \subset (-r-R, r+R)$.

   Let $x > r + r$, so

$$\begin{aligned} f * \varphi(x) &= \int_{-r}^{r} f(y)\varphi(x-y)dy \\ x > r + R &\Rightarrow x - y > r + R - r = R \Rightarrow \\ \varphi(x-y) &= \Rightarrow f * \varphi(x) = 0. \end{aligned}$$

   Similarly, if $x < -r - R$, we have $f * \varphi(x) = 0$.

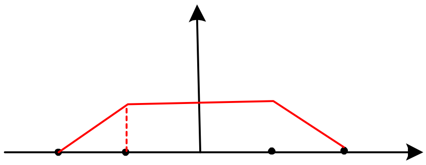
■

Lemma 2.3.1 *Suppose that $\rho \in L^1(\mathbb{R})$ and $v \in W^{1,p}(\mathbb{R})$. then $(\rho * v)$ is in $W^{1,p}(\mathbb{R})$ with $(\rho * v)' = \rho * v'$.*

Proof.

1. Suppose that ρ is of compact support and let $\varphi \in C_0^1(\mathbb{R})$

$$\begin{aligned} \int_{-\infty}^{\infty} (\rho * v)\varphi'(x) &= \int_{-k}^{k}\int_{-\infty}^{\infty} \rho(y)v(x-y)dy\varphi'(x)dx. \\ &= \int_{-k}^{k}\int_{-\infty}^{\infty} v(y)\rho(x-y)\varphi'(x)dydx \\ &= \iint (\rho(x-y)\varphi'(x)dx)v(y)dy \\ &= \int (\check{\rho} * \varphi)'(y)v(y)dy / \check{\rho}(s) = \rho(-s) \qquad \text{moreover } \check{\rho} * \varphi \in C_0^1(\mathbb{R}) \\ &= -\int v'(\check{\rho} * \varphi)(y)dy \text{ since } v \in W^{1,p}(\mathbb{R}) \\ &= -\int v'(y)\int \rho(x-y)\varphi(x)dxdy \\ &= -\iint \rho(x-y)\varphi(x)v'(y)dxdy \\ &= -\int \varphi(x)\int \rho(x-y)v'(y)dydx \\ &= -\int (\rho * v')(x)\varphi(x)dx, \quad \forall\, \varphi \in C_0^1(\mathbb{R}). \end{aligned}$$



Therefore,

$$\rho * v \in W^{1,p} \text{ with } (\rho * v)' = \rho * v'.$$

2. Suppose that $\rho \in L^1(\mathbb{R})$, then there exists a sequence $(\rho_n) \subset C_0^\infty(\mathbb{R})$ such that $\rho_n \longrightarrow \rho$ in L^1 norm.

   $\rho_n * v$ is in $W^{1,p}(\mathbb{R})$ such that $(\rho_n * v)' = \rho_n * v'$.

$$\|\rho_n * v - \rho * v\|_{L^p} = \|(\rho_n - \rho) * v\|_{L^p} \leq \|\rho_n - \rho\|_{L^1}\|v\|_{L^p}$$

   so as $n \longrightarrow \infty$, we get $\rho_n * v \longrightarrow \rho * v$ in L^p. Similarly, we get

$$\rho_n * v' \longrightarrow \rho * v' \text{ in } L^p.$$

   Therefore $\rho * v$ is in $W^{1,p}(\mathbb{R})$ with $(\rho * v)' = \rho * v'$.

Truncation. Let ξ be a $C_0^\infty(\mathbb{R})$ function with

$$\xi(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| \geq 2 \end{cases}$$

We define

$$\xi_n(x) = \xi\left(\frac{x}{n}\right), \quad n = 1, 2, 3, \ldots.$$

■

Lemma 2.3.2 *Let f be in $L^p(\mathbb{R})$, $1 \leq p < \infty$. Then $\xi_n f \longrightarrow f$ in L^p.*

Proof.

$$|\xi_n f - f|^p \longrightarrow 0 \text{ a.e. } x \in \mathbb{R}$$

$$|\xi_n f - f|^p \leq 2^{p-1}[|\xi_n f|^p + |f|^p] \leq 2|f|^p \in L^2.$$

So $\displaystyle\int_{\mathbb{R}} |\xi_n f - f|^p \longrightarrow 0$ by convergence dominate theorem. ■

Definition 2.3.3 (Molifiers) [see next page].

Theorem 2.3.2 (Density). *Let u be $W^{1,p}(I)$, $\quad 1 \leq p < \infty$, then there exists a sequence $(u_n) \subset C_0^\infty(\mathbb{R})$ such that $u_{n_{|_I}} \to u$ in $W^{1,p}(I)$.*

Proof. If $I \neq \mathbb{R}$ we then extend u on $\mathbb{R}$ by the extension operator P_u, so let us suppose that $I = \mathbb{R}$. We take $u_n = \xi_n(\rho_n * u)$ such that $\rho_n \subset C_0^\infty(\mathbb{R})$ and $\rho_n * u \to u$ in $L^p(\mathbb{R})$. $u_n \in C_0^\infty(\mathbb{R})$

$$\begin{aligned}\|u_n - u\|_{L^p} &= \|\xi_n(\rho_n * u) - u\|_{L^p} \\ &\leq \|\xi_n(\rho_n * u - u)\|_{L^p} + \|\xi_n u - u\|_{L^p} \\ &\leq \|\rho_n * u - u\|_{L^p} + \|\xi_n u - u\|_{L^p}.\end{aligned}$$

As $n \to \infty$, $\|u_n - u\|_{L^p} \to 0$.

$$u_n' = \xi_n'(\rho_n * u) + \xi_n(\rho_n * u').$$

So

$$\begin{aligned}\|u_n'\| - u'\|_{L^p} &\leq \|\xi_n'(\rho_n * u)\|_{L^p} + \|\xi_n(\rho_n * u') - u'\|_{L^p} \\ &\leq \frac{C}{n}\|u\|_{L^p} + \|\xi_n(\rho_n * u') - u'\|_{L^p}.\end{aligned}$$

As $n \to \infty$ $\|\xi_n(\rho_n * u') - u'\|_{L^p} \to 0$. So $\|u_n' - u'\|_{L^p} \to 0$ as $n \to \infty$. ■

Modifiers: Let

$$\psi(x) = \begin{cases} e^{-\frac{1}{(x-1)^2}}, & -1 < x < 1 \\ 0, & |x| \geq 1 \end{cases}$$
$$\psi \in C_0^\infty(\mathbb{R}).$$

We define the molifiers or regularizing sequences by $\psi_n = \dfrac{n}{a_0}\psi(nx)$, where $a_0 = \displaystyle\int_{\mathbb{R}} \psi(x)dx$. It is easy to verify that

1. $\displaystyle\int_{-\infty}^{\infty} \psi_n(x)dx = 1.$

2. supp $\psi_n \subset \left(-\frac{1}{n}, \frac{1}{n}\right)$, hence supp $\psi_n \to \{0\}$.

3. For $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} \psi_n(x)\varphi(x)dx - \varphi(0)\right| \\
&= \left|\int_{-\infty}^{\infty} \psi_n(x)[\varphi(x) - \varphi(0)]dx\right| \\
&= \left|\frac{1}{a}\int_{-1}^{1} \rho(y)\left[\varphi\left(\frac{y}{n}\right) - \varphi(0)\right]dy\right| \\
&\frac{1}{a}\sup_{[-1,1]}\left|\varphi\left(\frac{y}{n}\right) - \varphi(0)\right|\int_{-1}^{1}\rho(y)dy \\
&\le \sup\left|\varphi\left(\frac{y}{n}\right) - \varphi(0)\right| \stackrel{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Since φ is uniformly continuous on $[-1, 1]$ we conclude that $\psi_n \to \delta$ in $\mathcal{D}'(\mathbb{R})$.

2.4 Imbedding

Definition 2.4.1 Given two sets A and B. We say that A is imbedded in B; $A \subset B$; if $\forall\ a \in I$, $a \in B$.

Theorem 2.4.1 (Imbedding Theorem). *There exists a constant C, depending on $|I|$ such that*

$$\|u\|_{L^\infty(I)} \le C\|u\|_{W^{1,p}(I)}, \quad \forall\ u \in W^{1,p}(I).$$

Proof. Without loss of generality, we take $I = \mathbb{R}$; otherwise we extend u to $\tilde{u}$ on $\mathbb{R}$. So, let $G(s)\ |s|^{p-1}s$, $\quad p \ge 1$ and $w = G(v)$ for $v \in C_0^1(\mathbb{R})$. It is clear that $w \in C_0^1(\mathbb{R})$

and $w' = p|v|^{p-1}v'$. Thus

$$\begin{aligned} G(v(x)) &= \int_{-\infty}^{x} p|v|^{p-1}v'(t)dt \\ |v(x)|^{p-1}v(x) &= p\int_{-\infty}^{x} |v|^{p-1}v'(t)dt \Rightarrow \\ |v(x)|^p &\leq p\|v\|_{L^p}^{p-1}\|v'\|_{L^p}, \quad \forall\, x \in \mathbb{R}. \\ &\leq p\|v\|_{W^{1,p}(I)}^p, \quad \forall\, x \in \mathbb{R}. \end{aligned}$$

So

$$\|v\|_\infty \leq \sqrt[p]{p}\|v\|_{W^{1,p}(I)}.$$

If $u \in W^{1,p}(I)$, we know that $\exists\ (v_n) \subset C_0^\infty(\mathbb{R})$ such that $v_n \to u$ in $W^{1,p} \Rightarrow \|v_n\|_\infty \leq C\|v_n\|_{W^{1,p}(I)}$. As $n \to \infty$, $\|u\|_\infty \leq C\|u\|_{W^{1,p}(I)}$. ■

Remark 2.4.1 When $I = \ \mathbb{R}$, the imbedding constant $C = P^{1/p}$. If $I \neq R$, then the constant $C = C(|I|, P)$. This comes from the extension operator.

Theorem 2.4.2 (Compact Imbedding Theorem). *Suppose that I is a bounded interval, then*

a. The imbedding $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is compact for $\rho > 1$.

b. The imbedding $W^{1,1}(I) \hookrightarrow L^s(I)$ is compact for $s \in [1, +\infty)$.

Definition 2.4.2 We say that A is compactly imbedded in B, if any bounded subset of A has a convergent sequence in B. Or any bounded subset of A is relatively compact in B.

Proof.

(a) Let B be the unit ball in $W^{1,p}(I)$, $B = \{u \in W^{1,p}(I)/\|u\|_{W^{1,p}} \le 1\}$. For any x, y in I, we have

$$\begin{aligned} |u(x) - u(y)| &\le \left| \int_y^x u'(t)dt \right| \le \left(\int_I |u'|^p \right)^{1/p} |x-y|^{1/p} \\ &\le |x-y|^{1/p}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p > 1. \end{aligned}$$

Hence B is equicontinuous.

Also,

$$|u(x) - u(x_0)| \le |x - x_0|^{1/p'}, \qquad x_0 \in I$$

$$|u(x)| \le |u(x_0)| + |I|^{1/p'} \le C\|u\|_{W^{1,p}} + |I|^{1/p'}$$

$$\Rightarrow \|u\|_\infty \le C + |I|^{1/p'} = M$$

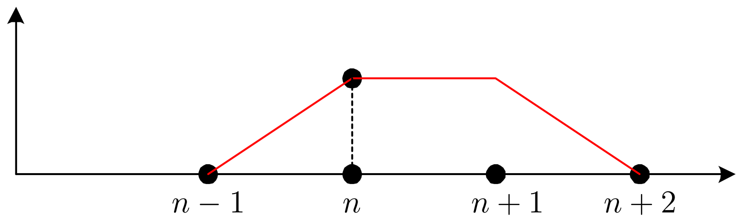
C is the embedding constant. Hence B is uniformly bounded. Arzela-Ascoli theorem shows that B is relatively compact.

b) To show that $W^{1,1}(I)$ is compactly imbedded in $L^s(I)$, $\quad s \ge 1$, we use a result of the L^p spaces. That is we show that

$$\|\tau_h u - u\|_{L^s(\omega)} \longrightarrow 0 \text{ uniformly in } u \text{ as } h \longrightarrow 0$$

where $u \in B$ and $\omega \subset\subset I$.

$$\begin{aligned} &\int_\omega |u(x+h) - u(x)|^s dx \\ &\quad = \int_\omega |u(x+h) - u(x)|^{s-1} |u(x+h) - u(x)| dx \\ &\quad \le 2\|u\|_\infty^{s-1} \int_\omega |u(x+h) - u(x)| dx \\ &\quad \le 2C\|u\|_{W^{2,1}(I)}^{r-1} \int_\omega \left| \int_x^{2+h} u'(t) \right| dt dx \\ &\quad \le 2C\|u'\|_{L^1(I)} \cdot |h| \\ &\quad \le C'\|u\|_{W^{1,1}(U)} |h|. \end{aligned}$$



■

Conclusion. $\|\tau_h u - u\|_{L^s} \leq C'|h|^{1/s} \to 0$; hence B is relatively compact in $L^s(I)$.

Remark 2.4.2 It is important that I is bounded in the compact imbedding

Example 2.4.1 Let

$$u_n(x) = \begin{cases} 1, & n \leq x \leq n+1 \\ 0, & x \leq n-1 \\ 0, & x \geq n+2 \end{cases}$$

be defined and continuous on $I = (0, +\infty)$.

It is easy to verify that $\|u_n\|_{W^{1,p}}$ is bounded and that $\lim_{n\to\infty} u_n(x) = 0$. This is a simple convergence; however, for any subsequence (u_{n_k}) we have

$$s \sup_{x\in\mathbb{R}} |u_{n_k}(x) - u(x)| = \sup_{x\in\mathbb{R}} |u_{n_k}(x)| = 1$$

Thus

$$\lim_{k\to\infty} \sup_{x\in\mathbb{R}} |u_{n_k}(x)| = 1 \neq 0.$$

Thus we cannot extract a subsequence, which converges to $u(x) \equiv 0$ in $C(\mathbb{R})$ uniformly.

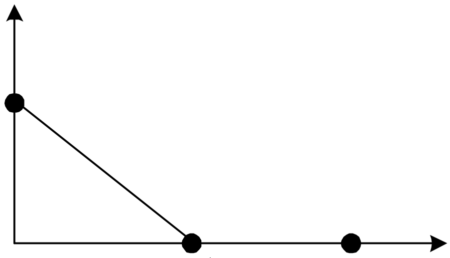
Also, note that

$$\begin{aligned} \|u_{n_k} - u\|_{L^s} &= \int_0^\infty u_{n_k}^s(x)ds \\ &\geq \int_{n_k}^{n_k+1} u_{n_k}^s(x)dx = 1. \end{aligned}$$

Hence,

$$\lim_{k\to\infty} \|u_{n_k} - u\|_{L^s} \geq 1, \quad \forall\ s \geq 1.$$

So the imbedding of $W^{1,1}(\mathbb{R})$ in $L^s(\mathbb{R})$ is not compact.



Remark 2.4.3 The imbedding of $W^{1,1}(I)$ in $C(I)$ is continuous but it never compact even if I is bounded.

Example 2.4.2 (This is not a proof).

$$\begin{aligned} u_n(x) &= \begin{cases} 1 - nx, & 0 < x \leq 1/n \\ 0, & 1 > x > 1/n \end{cases} \qquad n \geq 2 \\ \int_0^1 |u_n(x)| dx &= \frac{1}{2n} \geq 1 \\ \int_0^1 |u_n'(x)| dx &= \int_0^{1/n} n \, dx = 1. \end{aligned}$$

Note that

$$\int_0^1 |u_n'(x)|^p dx = n^{p-1} \to \infty \text{ as } n \to \infty, p > 1.$$

Thus (u_n) is bounded in $W^{1,1}(I)$ only.

$\lim\limits_{k\to\infty} u_{n_k}(x) = u(x) = 0$ for any subsequence. But $\sup\limits_I |u_{n_k}(x) - u(x)| = 1$. Hence u_{n_k} cannot converge uniformly to u.

Theorem 2.4.3 *Suppose that I is bounded or nonbounded. Then $W^{1,p}(I)$ is imbedded in $L^q(I)$, $\forall\ a \in [p, +\infty]$.*

Proof. $q = p$ or $q = +\infty$ is trivial. For $q \in (p, +\infty)$, we have

$$\int_I |u|^q dx \leq \|u\|_\infty^{q-p} \int_I |u|^p dx.$$

■

Remark 2.4.4 If I is bounded, then $q \in [1, +\infty)$. If I is unbounded then in general $q \in [p, +\infty)$.

Corollary 2.4.1 *Let $I = (a, +\infty)$ and $u \in W^{1,p}(I)$, $\quad 1 \leq p < \infty$. Then* $\lim_{x \to \infty} u(x) = 0$.

Proof. $u \in W^{1,p}(I) \Rightarrow$ there exists (u_n) in $C^\infty(\mathbb{R})$ such that $u_{n|I} \to u$ in $W^{1,p}(I) \Rightarrow u_{n|I} \to u$ in $L^\infty(I)$ from the imbedding

$$\begin{aligned} \Rightarrow |u(x)| &\leq |u(x) - u_n(x)| + |u_n(x)| \\ &< \epsilon + |u_n(x)|, \quad \text{for } n \text{ large} \\ &\Rightarrow \lim_{x \to \infty} |u(x)| < \epsilon + \lim_{x \to \infty} |u_n(x)| = \epsilon. \end{aligned}$$

Since ϵ is arbitrary, then $\lim_{x \to \infty} |u(x)| = 0$. ■

Remark 2.4.5

1. For $p = +\infty$, the assertion of the corollary is not true. Take $u(x) = 1$.

2. The $W^{1,p}$ functions do not oscillate at infinity.

Corollary 2.4.2 *If u and v are in $W^{1,p}(I)$, $\quad 1 \leq p \leq +\infty$, then uv in $W^{1,p}(I)$ and*

$$(uv)' = u'v + uv' \tag{$*$}$$

Moreover, for all x, y in $\overline{I}$, we have

$$\int_x^y (u'v + uv')(t)dt = u(y)v(y) - u(x)v(x). \tag{$**$}$$

Proof. $u, v \in W^{1,p}(I) \Rightarrow u, v \in L^\infty(I) \Rightarrow uv \in L^p(I)$ since $\int_I |uv|^p dx \leq \|u\|_\infty^p \|v\|_{L^p}^p$.
■

Case 1. $1 \leq p < \infty$.

Let (u_n) and (v_n) be two sequences in $C_0^\infty(\mathbb{R})$ such that $u_{n_I} \to u$ and $v_{n|I} \to v$ in $W^{1,p}(I)$; hence $u_{n|I} \to u$ and $v_{n|I} \to v$ in $L^\infty(I)$. Therefore $u_n v_n \to uv$ in $L^\infty(I)$. We also have

$$(u_n v_n)' = u_n' v_n + u_n v_n' \to u'v + uv' \text{ in } L^p(I).$$

$\Rightarrow (uv)' = u'v + uv' \in L^p(I)$; hence

$$uv \in W^{1,p}(I).$$

By integrating over (x, y):

$$\int_x^y (u_n v_n)' = \int_x^y (u_n' v_n + u_n v_n') = u_n(y)v_n(y) - u_n(x)v_n(x).$$

By letting $n \to \infty$, we obtain $(**)$.

Case 2. $p = +\infty$

$u, v \in W^{1,\infty}(I) \Rightarrow uv, u'v + uv' \in L^\infty(I)$. We have to verify that $(uv)' = uv' + u'v \in L^\infty(I)$. Let $\varphi \in C_0^1(I)$; so for J bounded and supp $\varphi \subset J \subset I$, we have u and $v \in L^q(J), \quad \forall\, q < \infty$ and consequently, by Case 1, we obtain

$$\int_I uv\varphi' = \int_J uv\varphi' = -\int_J (u'v + uv')\varphi = -\int_I (u'v + uv')\varphi.$$

Thus

$$(uv)' = uv' + uv' \text{ in } L^\infty(I).$$

This completes the proof.

Corollary 2.4.3 *Let $G \in C^1(\mathbb{R})$, such that $G(0) = 0$ and $u \in W^{1,p}(I)$. Then $G \circ u \in W^{1,p}(I)$ and $(G \circ u)' = (G' \circ u)u'$.*

Proof. $u \in W^{1,p}(I) \Rightarrow$ there exists $M > 0$ such that $-M \leq u(x) \leq M, \quad \forall\, x \in I$; $u \in C(\overline{I})$. G' is continuous and $G(0) = 0 \Rightarrow |G(s)| \leq C|s|, \quad \forall\, s \in [-M, M] \to G \circ u \in$

$L^p(I)$. Also $(G' \circ u)u' \in L^p(I)$, since $G' \circ u \in L^\infty$ and $u' \in L^p$. Now we should verify that $\int_I (G \circ u)\varphi' = -\int (G' \circ u)u'\varphi$ for all $\varphi \in C_0^1(I)$. ■

Case 1. $1 \le p < +\infty$.

There exists $(u_n) \in C_0^\infty(\mathbb{R})$ such that $u_{n|I} \to u$ in $W^{1,p}(I)$ and in $L^\infty(I)$. So $G \circ u_n \to G \circ u$ and $(G' \circ u_n)u_n' \to (G' \circ u)u'$ in $L^p(I)$ but

$$\int_I (G \circ u_n)\varphi' = -\int (G' \circ u_n)u_n'\varphi, \quad \forall\, \varphi \in C_0^\infty(I).$$

By taking n to ∞, we obtain

$$\int_I (G \circ u)\varphi' = -\int (G' \circ u)u'\varphi, \quad \forall\, \varphi \in C_0^\infty(I).$$

So $(G \circ u)' = (G' \circ u)u'$ by definition of weak derivative.

Case 2. $p + +\infty$.

We repeat the same analysis of the previous corollary.

Remark 2.4.6 The condition $G(0) = 0$ is not necessary when I is bounded.

Remark 2.4.7 $W^{1,p}(I)$ is called Banach algebra since $uv \in W^{1,p}(I)$ whenever u and v are in $W^{1,p}(I)$. This is not the case for $L^p(I)$ even for I bounded.

Example 2.4.3 $u(x) = \dfrac{1}{\sqrt{x}} \in L^1((0,1))$:

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2.$$

$u' \notin L^1(0,1)$ since $\displaystyle\int_0^1 u'(x)dx = u(x)\Big|_0^1 = \infty$. For $v = u, uv = \dfrac{1}{x}$,

$$\int_0^1 (uv)dx = \log x\Big|_0^1 = \infty \to uv \notin L^1((0,1)).$$

Example 2.4.4 Let $I = (0, +\infty)$ and $u \in L^p(I), \quad \forall\, 1 \le p < +\infty$. Take $G(s) = a \neq 0$; hence $G(0) \neq 0$. Note that $G \circ u = a \notin L^p(I)$.

2.5 Sobolev Spaces for Higher Orders

Definition 2.5.1 Given an integer $m \geq 2$, we define the Sobolev space of order m by

$$\begin{aligned} W^{m,p}(I) &= \left\{ u \in L^p(I)/u' \in W^{m-1,p}(I) \right\} \\ &= \left\{ u \in L^p(I)/u', u'', \ldots, u^{(m)} \in L^p(I) \right\}. \end{aligned}$$

Here the derivatives are in the weak sense.

We then see $H^m(I) = W^{m,2}(I)$.

1. It is easy to verify that $u \in W^{1,p}(I)$ if and only if there exists m functions $g_1, g_2, \ldots, g_m n \in L^p(I)$ such that

$$\int_I u\varphi^{(j)}(x)dx = (-1)^j \int_I g_j \varphi(x)dx, \quad \forall\ \varphi \in C_0^\infty(I).$$

2. The space $W^{1,p}(I)$ equipped with the norm:

$$\|u\|_{m,p} = \|u\|_p + \sum_{j=1}^{m} \|u^j\|_p$$

is a Banach space.

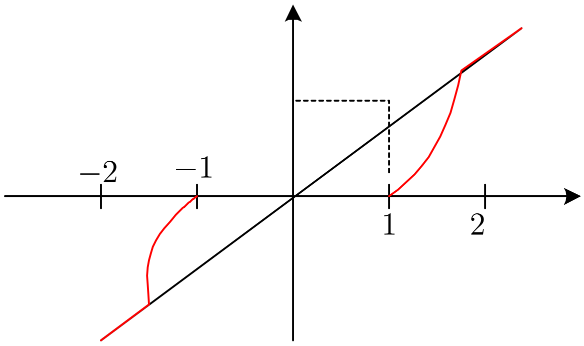
3. The space $H^m(I)$ equipped with the inner product

$$\langle u, v \rangle_{m,p} = \int_I (uv + u'v' + \cdots + u^{(m)}v^{(m)})(x)dx$$

is a Hilbert space.

Theorem 2.5.1 *$W^{1,p}(I)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.*

Theorem 2.5.2 *$W^{m,p}(I)$ is continously imbedded in $C^{m-1}(\overline{I})$.*



2.6 The Space $W_0^{1,p}(I)$

Definition 2.6.1 For $1 \leq p < +\infty$, we define $W_0^{1,p}(I)$ to be the closure of $C_0^1(I)$ with respect to the norm of $W^{1,p}(I)$. We denote by $H_0^1(I) = W_0^{1,2}(I)$.

Properties:

1. It is clear that $W_0^{1,p}(I)$ is a Banach space if equipped with the norm of $W^{1,p}(I)$.

2. $H_0^1(I)$ is a Hilbert space with respect to the inner product $\langle u, v \rangle = \displaystyle\int_I (uv + ou'v')(x)dx$.

3. $W_0^{1,p}(I)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < +\infty$.

4. In the case $I = \mathbb{R}$, $W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R})$ since $C_0^1(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, $1 \leq p < +\infty$.

Remark 2.6.1 $C_0^1(I)$ (or $C_0^\infty(\mathbb{R})$) is not dense in $L^\infty(I)$; otherwise all L^∞ functions are continuous.

Theorem 2.6.1 *Suppose that $u \in W^{1,p}(I)$. Then $u \in W_0^{1,p}(I)$ if and only if $u = 0$ on ∂I (boundary of I).*

Proof.

1. Given u in $W_0^{1,p}(I)$, so u is in $C(\overline{I})$. We know there exists a sequence (u_n) in $C_0^1(I)$ such that $u_n \to u$ in $W^{1,p}(I)$, hence in L^∞ norm. $\forall\ \epsilon > 0$, there exists $n \in \mathbb{N}$ such that $|u_n(x) - u(x)| < \epsilon, \quad \forall\ x \in \overline{I}$; in particular for $x \in \partial I$ we have $|u(x)| < \epsilon$ since $u_n(x) = 0, \quad x \in \partial I$. Thus $\forall\ \epsilon > 0, |u(x)| < \epsilon, \quad x \in \partial I \Rightarrow u = 0$ on ∂I.

2. Let G be a C^1 function on $\mathbb{R}$ such that

$$\begin{aligned} G(s) &= \begin{cases} 0, & \delta \leq 1 \\ s, & s \geq 2 \end{cases} \\ |G(s| &\leq |s| \\ |G'(s)| &\leq k. \end{aligned}$$

Given u in $W^{1,p}(I)$ such that $u_{|\partial I} = 0$. Define $u_n(x) = \dfrac{1}{n}G(nu(x))$. It is easy to verify that $u_n \in W^{1,p}(x)$ since $|u_n| \leq |u|$ and $u_n'(x) = G'(nu(x))u'(x)$. So $|u_n'| \leq k|u'|$. Also, supp $u_n \subset \left\{x \in I/|u(x)| \geq \dfrac{1}{n}\right\}$, which is a compact set of I since $\lim\limits_{\text{“}x\to\partial I\text{”}} u(x) = 0$. Thus

$$u_n \in W^{1,p}(I) \cap C_0(I) \Rightarrow u_n W_0^{1,p}(I).$$

Next, we prove that $u_n \to u$ in $W^{1,p}(I)$. For this we use the dominated convergence theorem.

First, note that $|u_n - u| \leq 2|u|$ and

$$|u_n' - u'| \leq (k+1)|u'|.$$

It is easy to verify that $|u_n - u| \to 0$ and $|u_n' - u'| \to 0$ for almost every $x \in I$ and since $2|u| \in L^p$ and $(k+1)|u'| \in L^p(I)$. Then $|u_n - u| \to 0$ and $|u_n' - u| \to 0$ in $L^p(I)$; hence $u_n \to u$ in $W^{1,p}(I)$. Since $u_n \in W^{1,p}(I)$, which is a closed subspace of $W^{1,p}(I)$, then $u \in W^{1,p}(I)$. ■

Theorem 2.6.2 *1. Let $1 < p < \infty$ and $u \in L^p(I)$. Then $u \in W_0^{1,p}(I)$ if and only if there exists a constant C such that*

$$\left|\int_I u\varphi'\right| \le C\|\varphi\|_{L^{p'}(I)}, \quad \forall\, \varphi \in C_0^1(I), \frac{1}{p}+\frac{1}{p'}=1.$$

2. Let $1 \le p < \infty$ and $u \in L^p(I)$. We define $\tilde{u}$ by

$$\tilde{u}(x) = \begin{cases} u(x), & x \in I \\ 0, & x \in \mathbb{R}\setminus I \end{cases}$$

Then $u \in W_0^{1,p}(I)$ if and only if $\tilde{u} \in W^{1,p}(\mathbb{R})$.

Proof.

1. $u \in W_0^{1,p}(I) \Rightarrow \left|\int_I u\varphi'\right| = \left|-\int_I u'\varphi\right|, \quad \forall\, \varphi \in C_0^1(I)$

$$\Rightarrow \left|\int_I u\varphi'\right| \le \|u'\|_{L^p}\|\varphi\|_{L^{p'}}, \quad \forall\, \varphi \in C_0^1(I), \frac{1}{p}+\frac{1}{p'}=1.$$

$*$Define the linear form on $C_0^1(I)$ by $F(\varphi) = -\int_i u\varphi'$.

It is clear that $\|F(\varphi)\| \le C\|\varphi\|_{L^{p'}} \to F$ is bounded on a subspace of $L^p(I)$. So it can be extended to $\tilde{F}$ which is bounded by the same constant C on $L^{p'}(I)$. Riesz Representation Theorem implies the existence of $g \in L^p(I)$ such that

$$F(\varphi) = \int_I g\varphi', \quad \forall\, \varphi \in L^{p'}(I).$$

Thus (by definition) $g = u'$

$$\Rightarrow u \in W^{1,p}(I).$$

To prove that $u \in W_0^{1,p}(I)$, we use the fact that $g(\varphi) = -\int_I u\varphi' = \int_I u'\varphi$ for all $\varphi \in C^1(I) \subset L^{p'}(I)$

$$\begin{aligned} -\int_a^b u'\varphi &= -u(b)\varphi(b) + u(a)\varphi(a) + \int_I u\varphi' \Rightarrow \\ u(a)\varphi(a) &= u(b)\varphi(b), \quad \forall\, \varphi \in C^1(\overline{I}) \Rightarrow u(a) = u(b) = 0 \end{aligned}$$

$$\Rightarrow \quad u \in W_0^{1,p}(I).$$

Note that we needed $1 < p < \infty$ for the reflexivity of $L^{p'}(I)$ and hence we could apply the Riesz Representation Theorem.

2. $u \in W^{1,p} \Rightarrow \tilde{u} \in L^p(\mathbb{R})$ since $\displaystyle\int_{\mathbb{R}} |\tilde{u}|^p = \int_I |u|^p$. Let $\varphi \in C_0^1(\mathbb{R})$. So

$$\begin{aligned} \int_{\mathbb{R}} \tilde{u}\varphi &= \int_a^b u\varphi' = u(b)\varphi'(b) - u(a)\varphi'(b) - \int_a^b u'\varphi \\ &= -\int_a^b u'\varphi \text{ since } u(a) = u(b) = 0 \\ &= \int_{\mathbb{R}} \tilde{u}'\varphi, \quad \tilde{u}' = \begin{cases} u' & \text{on } I \\ 0, & \text{on } \mathbb{R} \setminus I \end{cases} \end{aligned}$$

$\tilde{u}' \in L^p(R)$

*since $\tilde{u} \in W^{1,p}(I) \rightarrow \tilde{u}$ is continuous $\Rightarrow \tilde{u}(a) = \tilde{u}(b) = 0$.

To show that $u \in W^{1,p}(I)$, we take $\varphi \in C_0^1(I)$ and compute

$$\int_I u\varphi' = \int_{\mathbb{R}} \tilde{u}\tilde{\varphi}' = -\int_{\mathbb{R}} \tilde{u}'\tilde{\varphi} = -\int_I g\varphi$$

where $\tilde{\varphi}$ is the extension of φ, which belongs to $C_0^1(\mathbb{R})$ supp $\tilde{\varphi} =$ supp $\varphi \subset\subset \mathbb{R}$. Thus

$$u \in W^{1,p}(I) \Rightarrow u \in C(\overline{I}) \Rightarrow u(a) = \tilde{u}(a) = 0$$

and

$$u(b) = \tilde{u}(b) = 0 \Rightarrow u \in W_0^{1,p}(I).$$

■

Theorem 2.6.3 (Poincar's inequality). *Suppose that I is bounded. Then there exists a constant*

$$C = C(I) > 0 \textit{ such that } \int_0 |u|^p \leq C \int_I |u'|^p, \quad \forall\, u \in W_0^{1,p}(I).$$

Proof.

$$\begin{aligned} u \in W_0^{1,p}(I) &\Rightarrow u(x) = u(a) + \int_a^x u'(t)dt = \int_a^x u(t)dt \\ &\Rightarrow |u(x)|^p \le \left(\int_a^x |u'(t)|dt \right)^p = \left(\int_a^x 1 \cdot |u'(t)|dt \right)^p \\ &\le \left[(x-a)^{1/p'} \left(\int_a^b |u'(t)|^p dt \right)^{1/p} \right]^p \\ &\le (x-a)^{p/p'} \|u'\|_{L^p(I)}^p, \frac{1}{p} + \frac{1}{p'} = 1 \end{aligned}$$

integrate over (a, b):

$$\int_I |u(x)|^p dx \le \|u'\|_{L^p(I)}^p \cdot \frac{|I|^p}{p}.$$

Therefore

$$\|u\|_{L^p(I)} \le \frac{|I|}{\sqrt[p]{p}} \|u'\|_{L^p(I)}.$$

■

Remark 2.6.2 1. From the above inequality it is important that I is bounded.

2. For $p = +\infty$, we have $C = |I|$, that is $\lim_{p\to\infty} \sqrt[p]{p} = 1$.

3. Looking carefully into the proof, we easily see that Poincare's inequality holds for

$$u \in W^{1,p}(I) \text{ with } u(c) = 0, \quad a \le c \le b < \infty.$$

Corollary 2.6.1 *The quantity $\|u'\|_{L^p}$ define an "equivalent" norm on $W_0^{1,p}(I)$ and $\langle u, v\rangle = \int_a^b u'v'$ define an "equivalent" inner product on $H_0^1(I)$.*

2.7 The Space $W_0^{m,p}(I)$

Definition 2.7.1 Let $1 \le p < +\infty$, we define $W_0^{m,p}(I)$ to be the closure of $C_0^m(I)$ (or $C_0^\infty(I)$) with respect to the norm of $W^{m,p}(I)$. We denote by $H_0^m(I) = W_0^{m,2}(I)$.

Remark 2.7.1 All the properties of $W_0^{1,p}(I)$ hold for $W_0^{m,p}(I)$.

Proposition 2.7.1 *For*

$$1 \leq p < \infty, W_0^{1,p}(I) = \left\{ u \in W^{m,p}(I) / u_{|\partial I} = u'_{|\partial I} = \cdots = u^{(m-1)}_{|\partial I} = 0 \right\}$$

Remark 2.7.2 There are 2 important spaces, namely

$$W^{2,p}(I) = \{ u \in W^{2,p}(I) / u_{|\partial I} = u'_{|\partial I} = 0 \}$$

and

$$W^{2,p}(I) \cap W_0^{1,p}(I) = \{ u \in W^{2,p}(I) / u_{|\partial I} = 0 \}$$

Example 2.7.1 On $I = (0, \pi)$, let $u(x) = \sin x$. It is clear that $u \in W^{m,p}(I), \quad \forall\, m \geq 2, \quad p \geq 1 \qquad u(0) = u(\pi) = 0$; but $u'(0)u'(\pi) \neq 0$. So $u \notin W_0^{2,p}(I)$ however $u \in W^{2,p}(I) \cap W_0^{1,p}(I)$.

2.8 Dual space of $W_0^{1,p}(I)$

Definition 2.8.1