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Blow-up of Solutions of a Semilinear Heat Equation with a Visco-elastic Term

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Abstract. In this work we consider an initial boundary value problem related to the equation

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = |u|^{p-2}u$$

and prove, under suitable conditions on g and p, a blow-up result for solutions with negative or vanishing initial energy. This result improves an earlier one by the author.

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1. Introduction

In this work we study the finite-time blow-up of solutions for the following initial boundary value problem

$$\begin{aligned} u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds &= |u|^{p-2}u, \quad x \in \Omega, \quad t > 0 \\ u(x,t) &= 0, \quad x \in \partial\Omega, \quad t \geq 0 \\ u(x,0) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{1.1}$$

where $g : R_+ \to R_+$ is a bounded C^1 function, $p > 2$, and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$.

This type of equations arises from a variety of mathematical models in engineering and physical sciences. For example, in the study of heat conduction in materials with memory, the classical Fourier's law of heat flux is replaced by the following form

$$q = -d\nabla u - \int_{-\infty}^t \nabla\left[k(x,t)u(x,\tau)\right]d\tau, \tag{1.2}$$

where u is the temperature, d the diffusion coefficient and the integral term represents the memory effect in the material. The study of this type of equations has drawn a considerable attention see [3], [4], [12] , [14] [15]. From a mathematical point of view, one would expect the integral term to be dominated by the leading term in the equation. Therefore, the theory of parabolic equations applies to this type of equations.

In the absence of the memory term ($g = 0$) problem (1.1) has been studied by various authors and several results concerning global and nonglobal existence have been established. For instance, in the early 1970's Levine [8] introduced the concavity method and showed that solutions with negative energy blow up in finite time. Later, this method was improved by Kalantarov and Ladyzhenskaya [7] to accommodate more general situations. Ball [2] also studied (1.1) with $f(u, \nabla u)$ instead of $|u|^{p-2}u$ and established a nonglobal existence result in bounded domains. This result had been extended to unbounded domains by Alfonsi and Weissler [1].

For the quasilinear case, Junning [6] studied

$$\begin{aligned} u_t - \operatorname{div}(|\nabla u|^{m-2}\nabla u) &= f(u), \quad x \in \Omega, \quad t > 0 \\ u(x,t) &= 0, \quad x \in \partial\Omega \quad t \geq 0 \\ u(x,0) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{1.3}$$

and established a global existence result. He also proved a nonglobal existence result under the condition

$$\frac{1}{m}\int_\Omega |\nabla u_0(x)|^m dx - \int_\Omega F(u_0(x))dx \leq -\frac{4(m-1)}{mT(m-2)^2}\int_\Omega u_0^2(x)dx \tag{1.4}$$

where $F(u) = \int_0^u f(s)ds$. More precisely he showed that if there exists $T > 0$, for which (1.4) holds then the solution blows up in a time less than T. This type of results have been extensively generalized and improved by Levine, Park, and Serrin in a paper [9], where the authors proved some global, as well as nonglobal, existence theorems. Their result, when applied to problem (1.3), requires that

$$\frac{1}{m}\int_\Omega |\nabla u_0(x)|^m dx - \int_\Omega F(u_0(x))dx < 0. \tag{1.5}$$

We note that the inequality (1.5) implies (1.4). In a note, Messaoudi [10] extended the blow-up result to solution with initial datum satisfying

$$\frac{1}{m}\int_\Omega |\nabla u_0(x)|^m dx - \int_\Omega F(u_0(x))dx \leq 0. \tag{1.6}$$

In the present work, we consider (1.1) and prove, for suitable conditions on p and g, a blow-up result for solutions with negative or vanishing initial energy. This result improves an earlier one in [10].

2. Blow-up

In order to state and prove our result we introduce the "modified" energy functional

$$E(t) = \frac{1}{2}(g \diamond \nabla u)(t) + \frac{1}{2}(1 - \int_0^t g(s)ds) \|\nabla u(t)\|_2^2 - \frac{1}{p} \|u(t)\|_p^p \tag{2.1}$$

where

$$(g \diamond v)(t) = \int_0^t g(t-\tau) \|v(t) - v(\tau)\|_2^2 \, d\tau. \tag{2.2}$$

For the relaxation function g and p, we assume that

$$g(s) \geq 0, \qquad g'(s) \leq 0, \qquad 1 - \int_0^\infty g(s)ds = l > 0 \tag{2.3}$$

and

$$2 < p \leq \frac{2(n-1)}{n-2}, \ n > 2, \qquad p > 2, \quad n = 1, 2. \tag{2.4}$$

By multiplying the equation in (1.1) by u_t, integrating over Ω we get, after some manipulations, see [11],

$$\frac{d}{dt}E(t) = -\left(\frac{1}{2}g(t) \|\nabla u(t)\|_2^2 - \frac{1}{2}(g^{'} \diamond \nabla u)(t) + \int_\Omega |u_t|^2 u_t dx\right) \leq 0, \tag{2.5}$$

for regular solutions. The same result can be established, for weaker solutions and for almost every t, by a simple density argument.

Similarly to [13], we give a definition for a strong solution of (1.1).

Definition: A strong solution of (1.1) is a function $u \in C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega))$, satisfying (2.5) and

$$\int_0^t \int_\Omega \left(\nabla u . \nabla \phi - \int_0^s \nabla u(\tau) . \nabla \phi(s) d\tau + u_t \phi - |u|^{p-2} u\phi\right) dxds = 0,$$

for all t in $[0,T)$ and all ϕ in $C([0,T), H_0^1(\Omega))$.

Remark 2.1: Condition (2.4) is needed so that $|u|^{p-2}u \in L^2(\Omega)$; hence $\int_\Omega |u|^{p-2} u\phi dx$ makes sense. Condition $1 - \int_0^\infty g(s)ds = l > 0$ is necessary to guarantee the parabolicity of the system (1.1).

Theorem. *Assume that* (2.3) *and* (2.4) *hold. Given* $u_0 \in H_0^1(\Omega)$ *satisfying* $E(0) \leq 0$. *If*

$$\int_0^\infty g(s)ds < \frac{p-2}{p-3/2} \tag{2.6}$$

then any strong solution of (1.1) *blows up in finite time.*

Proof. We define

$$\mathcal{L}(t) = \frac{1}{2}\int_\Omega u^2(x,t)dx$$

and differentiate with respect to t to get

$$\begin{aligned}
\mathcal{L}'(t) &= \int_\Omega u u_t(x,t)dx \\
&= \int_\Omega u\Delta u dx - \int_\Omega u(x,t)\int_0^t g(t-s)\Delta u(x,s)dsdx + \int_\Omega |u|^p dx \\
&= -\int_\Omega |\nabla u|^2 dx + \int_\Omega \int_0^t g(t-s)\nabla u(x,t).\nabla u(x,s)dsdx + \int_\Omega |u|^p dx \\
&\geq -\int_\Omega |\nabla u|^2 dx + \int_0^t g(t-s)||\nabla u(t)||_2^2 d\tau + \int_\Omega |u|^p dx \qquad (2.7)\\
&\quad -\int_0^t g(t-s)\int_\Omega |\nabla u(t).[\nabla u(s) - \nabla u(t)]|dxd\tau.
\end{aligned}$$

By using the Schwarz inequality, (2.7) takes the form

$$\begin{aligned}
\mathcal{L}'(t) &\geq \int_\Omega |u|^p dx - (1 - \int_0^t g(s)ds)||\nabla u(t)||_2^2 \qquad (2.8)\\
&\quad -\int_0^t g(t-\tau)||\nabla u(t)||_2||\nabla u(\tau) - \nabla u(t)||_2 d\tau
\end{aligned}$$

By applying Young's inequality to the last term of (2.8) we arrive at

$$\mathcal{L}'(t) \geq \int_\Omega |u|^p dx - \left[1 - \frac{3}{4}\int_0^t g(s)ds\right]||\nabla u(t)||_2^2 - (g \diamond \nabla u)(t) \qquad (2.9)$$

We then substitute for $||\nabla u(t)||_2^2$ from (2.1); hence (2.9) becomes

$$\begin{aligned}
\mathcal{L}'(t) &\geq \int_\Omega |u|^p dx + 2\frac{\left[1 - \frac{3}{4}\int_0^t g(s)ds\right]}{(1 - \int_0^t g(s)ds)} H(t) \\
&\quad + \left(\frac{1 - \frac{3}{4}\int_0^t g(s)ds}{(1 - \int_0^t g(s)ds)} - 1\right)(g \diamond \nabla u)(t) \qquad (2.10)\\
&\quad -\frac{2}{p}\frac{1 - \frac{3}{4}\int_0^t g(s)ds}{(1 - \int_0^t g(s)ds)}\int_\Omega |u|^p dx \geq \gamma \int_\Omega |u|^p dx
\end{aligned}$$

where

$$\gamma = 1 - \frac{2}{p}\frac{1 - \frac{3}{4}\int_0^\infty g(s)ds}{(1 - \int_0^\infty g(s)ds)} > 0$$

because of (2.6). Next we have, by the embedding of the L^q spaces,

$$\mathcal{L}^{p/2}(t) \leq C||u||_p^p. \qquad (2.11)$$

By combining (2.10) and (2.11) we get

$$\mathcal{L}'(t) \geq \Gamma L^{p/2}(t), \qquad (2.12)$$

A direct integration of (2.12) then yields

$$\mathcal{L}^{p/2-1}(t) \geq \frac{1}{\mathcal{L}^{1-p/2}(0) - \Gamma t}.$$

Therefore $\mathcal{L}$ blows up in a time

$$t^* \leq \frac{1}{\Gamma \mathcal{L}^{(p/2)-1}(0)}. = \frac{1}{\Gamma}\left(\frac{1}{2}\int_\Omega u_0^2(x)dx\right)^{1-p/2}$$

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