Global existence and decay of solutions to a system of Petrovsky

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Abstract

In this paper we consider the nonlinearly damped semilinear Petrovsky equation

$$u_{tt} + \Delta^2 u + a u_t |u_t|^{m-2} = b u |u|^{p-2}$$

in a bounded domain, where $a, b > 0$. We prove, for a suitably chosen initial data, a global existence as well as a decay result.
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1 Introduction

In [5] we considered the problem

$$\begin{array}{c} u_{tt} + \Delta^2 u + a u_t |u_t|^{m-2} = b u |u|^{p-2}, \qquad x \in \Omega, \qquad t > 0 \\ u(x,t) = \partial_\nu u(x,t) = 0, \quad x \in \partial\Omega\ ,\ t \geq 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega, \end{array} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$, ν is the unit outer normal on $\partial\Omega$, a, b are strictly positive constants,and p, $m > 2$. We proved a local existence result 'of weak solutions' then showed that this solution continues to exist globally if $m \geq p$, however it blows up in finite time if $m < p$ and the initial energy is negative. In [2], Guesmia considered the following problem

$$\begin{array}{c} u_{tt}(x,t) + \Delta^2 u(x,t) + q(x)u(x,t) + g(u_t(x,t)) = 0, \qquad x \in \Omega, \qquad t > 0 \\ u(x,t) = \partial_\nu u(x,t) = 0, \quad x \in \partial\Omega\ ,\ t \geq 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega, \end{array} \tag{1.2}$$

where g is continuous, increasing, satisfying $g(0) = 0$, and $q : \Omega \to \mathbb{R}^+$, is a bounded function and proved a global existence and a regularity result. He also established, under suitable growth conditions on g, decay results for weak, as well as strong, solutions. Precisely, he showed that the solution decays exponentially if g behaves like a linear function, whereas the decay is of a polynomial order otherwise. Similar results to the system (1.2), coupled with a semilinear wave equation, have been established by Guesmia [3]. Also the system composed of the equation (1.2), with $\Delta^2 u_t(x,t) + \Delta g(\Delta u(x,t))$ in the place of $q(x)u(x,t) + g(u_t(x,t))$, has been treated by Aassila and Guesmia [1] and an exponential decay theorem, by using an important lemma of Komornik [4], has been established.

In this work we extend the decay result of [2] to (1.1) for initial data suitably chosen. Our argument of proof will follow closely the steps of [7] with the necessary modification needed. It worth mentioning that it is only for simplicity, q is taken to be zero, $g(u_t(x,t)) = au_t|u_t|^{m-2}$, and the source term has a power form. The same theorems could be established for more general functions.

2 Preliminaries

In this section we state, without proof, a local existence result for (1.1). For the proof we refer the reader to [5]. We also present some known results, which will be used later on in the paper.
bfProposition 2.1 *Assume that*

$$2 < p \leq 2(n-2)/n-4, \qquad 2 \leq m \leq 2n/(n-4), \tag{2.1}$$

for $n \geq 5$, *hold. Then given any* u_0 *in* $H_0^2(\Omega)$, *and* u_1 *in* $L^2(\Omega)$, *the problem (1.1) has a unique weak solution*

$$\begin{aligned} u &\in C\Big([0,\ T];\ H_0^2(\Omega)\Big) \\ u_t &\in C\Big([0,\ T];\ L^2(\Omega)\Big) \cap L^m\left(\Omega \times (0,\ T)\right), \end{aligned} \tag{2.2}$$

for some $T > 0$.

bfLemma 2.2 (Nakao [6]). Let $\phi(t)$ be a noninreasing and nonnegative function defined on $[0,T], T > 1$, satisfying

$$\phi^{1+r}(t) \leq k_0(\phi(t) - \phi(t+1)), \qquad t \in [0,T],$$

for $k_0 > 1$ and $r \geq 0$. Then we have, for each $t \in [0,T]$,

$$\begin{aligned} \phi(t) &\leq \phi(0)e^{-k[t-1]^+}, \qquad r = 0 \\ \phi(t) &\leq \{\phi(0)^{-r} + k_0 r[t-1]^+\}^{-1/r}, \qquad r > 0, \end{aligned}$$

where $[t-1]^+ = \max\{t-1, 0\}$ and $k = \ln(k_0/(k_0-1))$.

Also by using Poincare's inequality, there exists a constant C_1depending on Ω only such that for any $1 \leq q < \infty$, if $n \leq 4$ and for any $1 \leq q < 2n/(n-4)$, if $n \geq 5$, we have

$$||v||_q \leq C_* ||\Delta v||_2, \quad \forall v \in H_0^2(\Omega) \tag{2.3}$$

3 Global Existence

In order to state and prove our main results, we first introduce the following

$$\begin{aligned} I(t) &= I(u(t)) = ||\Delta u(t)||_2^2 - b||u(t)||_p^p \\ J(t) &= J(u(t)) = \frac{1}{2}||\Delta u(t)||_2^2 - \frac{b}{p}||u(t)||_p^p \\ E(t) &= E(u(t), u_t(t)) = J(t) + \frac{1}{2}||u_t(t)||_2^2 \\ H &= \{w \in H_0^2(\Omega) \quad / \quad I(w) > 0\} \cup \{0\} \end{aligned} \tag{3.1}$$

where we use $w(t)$ instead of $w(.,t)$.
bfRemark 3.1 By multiplying equation (1.1) by u_t, integrating over Ω, and using integration by parts and the boundary conditions we get

$$E'(t) = -||u_t(t)||_m^m \leq 0, \qquad \forall t \in [0,T). \tag{3.2}$$

bfLemma 3.1 *Suppose that (2.1) holds. If* $u_0 \in H$, *and* $u_1 \in L^2(\Omega)$ *such that*

$$bC_*^p \left(\frac{2p}{p-2}E(u_0,u_1)\right)^{(p-2)/2} < 1 \tag{3.3}$$

then the solution $u(t) \in H$, *for each* $t \in [0,T)$.
bfProof. Since $I(u_0) > 0$ then there exists $T_m \leq T$ such that $I(u(t)) \geq 0$ for all $t \in [0,T_m)$. This implies

$$\begin{aligned} J(t) &= \frac{1}{2}||\Delta u(t)||_2^2 - \frac{b}{p}||u(t)||_p^p \\ &= \frac{p-2}{2p}||\Delta u(t)||_2^2 + \frac{1}{p}I(u(t)) \\ &\geq \frac{p-2}{2p}||\Delta u(t)||_2^2, \qquad \forall t \in [0,T_m); \end{aligned} \tag{3.4}$$

hence

$$\begin{aligned} ||\Delta u(t)||_2^2 &\leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \\ &\leq \frac{2p}{p-2}E(u_0,u_1), \qquad \forall t \in [0,T_m). \end{aligned} \tag{3.5}$$

By exploiting (2.3), (3.3), and (3.5), we easily arrive at

$$\begin{aligned} b||u(t)||_p^p &\leq bC_*^p||\Delta u(t)||_2^p = bC_*^p||\Delta u(t)||_2^{p-2}||\Delta u(t)||_2^2 \\ &\leq bC_*^p\left(\frac{2p}{p-2}E(u_0,u_1)\right)^{(p-2)/2}||\Delta u(t)||_2^2 \\ &< ||\Delta u(t)||_2^2, \qquad \forall t \in [0,T_m); \end{aligned} \tag{3.6}$$

hence $||\Delta u(t)||_2^2 - b||u(t)||_p^p > 0$, $\forall t \in [0, T_m)$. This shows that $u(t) \in H, \forall t \in [0, T_m)$. By repeating the procedure, T_m is extended to T.
bfTheorem 3.2 *Suppose that (2.1) holds. If $u_0 \in H$, and $u_1 \in L^2(\Omega)$ satisfying (3.3) Then the solution is global*
bfProof. It suffices to show that $||\Delta u(t)||_2^2 + ||u_t(t)||_2^2$ is bounded independently of t. To achieve this we use (3.1) and (3.2)

$$\begin{aligned} E(u_0, u_1) &\geq E(t) = \frac{1}{2}||\Delta u(t)||_2^2 - \frac{b}{p}||u(t)||_p^p + \frac{1}{2}||u_t(t)||_2^2 \\ &= \frac{p-2}{2p}||\Delta u(t)||_2^2 + \frac{1}{p}I(u(t)) + \frac{1}{2}||u_t(t)||_2^2 \\ &\geq \frac{p-2}{2p}||\Delta u(t)||_2^2 + \frac{1}{2}||u_t(t)||_2^2 \end{aligned} \tag{3.7}$$

since $I(u(t)) \geq 0$. Therefore

$$||\Delta u(t)||_2^2 + ||u_t(t)||_2^2 \leq C_1 E(u_0, u_1) \tag{3.8}$$

for $C = \max\{2, 2p/(p-2)\}$.
bfRemark 3.2 If $m \geq p$ then the global existence can be obtained for any $u_0 \in H_0^2(\Omega)$ and any $u_1 \in L^2(\Omega)$. See [5].

4 Asymptotic Behavior

Before stating and proving the decay result, we start with
bfLemma 4.1 *Suppose that (2.1) holds. If $u_0 \in H$, and $u_1 \in L^2(\Omega)$ satisfying (3.3). Then*

$$b||u(t)||_p^p \leq (1-\eta)||\Delta u(t)||_2^2 \tag{4.1}$$

where $\eta = 1 - bC_*^p \left(\frac{2p}{p-2}E(u_0, u_1)\right)^{(p-2)/2}$.
bfProof. It suffices to rewrite (3.6) as

$$b||u(t)||_p^p \leq \left\{1 - \left[1 - bC_*^p\left(\frac{2p}{p-2}E(u_0, u_1)\right)^{(p-2)/2}\right]\right\}||\Delta u(t)||_2^2.$$

bfTheorem 4.2 *Suppose that (2.1) holds. If $u_0 \in H$, and $u_1 \in L^2(\Omega)$ satisfying (3.3). Then the solution satisfies the following decay estimates :*

$$\begin{aligned} E(t) &\leq E(u_0, u_1)e^{-k[t-1]^+}, \quad t \geq 0, \quad m = 2 \qquad (4.2) \\ E(t) &\leq \left(E(u_0, u_1)]^{-(m-2)/2} + \frac{(m-2)C}{2}[t-1]^+\right)^{-2/(m-2)}, \quad t \geq 0, \quad m > 2, \end{aligned}$$

where $[t-1]^+ = \max\{t-1, 0\}$ and k, C are constants depending on $||\Delta u_0||_2$ and $||u_1||_2$.
bfProof We multiply equation (1.1) by u_t and integrate over $\Omega \times [t, t+1]$ to get

$$E(t) - E(t+1) = a\int_t^{t+1} ||u_t(s)||_m^m ds = a[F(t)]^m. \tag{4.3}$$

We then use Holder's inequality to estimate

$$\begin{aligned}\int_t^{t+1} ||u_t(s)||_2^2 ds &= \int_t^{t+1}\int_\Omega |u_t(x,s)|^2 dxds \qquad (4.4)\\ &\le C(\Omega)\left(\int_t^{t+1} ||u_t(s)||_m^m ds\right)^{2/m} \le C(\Omega)[F(t)]^2.\end{aligned}$$

By applying the mean value theorem to the lefthand side of (4.4), we obtain

$$||u_t(t_i)||_2 \le 2C(\Omega)F(t), \quad i = 1, 2,$$

for some $t_1 \in [t,\, t+1/4]$ and $t_2 \in [t+3/4,\, t+1]$. Next we multiply equation (1.1) by u and integrate over $\Omega \times [t_1,\, t_2]$ to arrive at

$$\begin{aligned}\int_{t_1}^{t_2} I(s)ds &\le \sum_{i=1}^{2} ||u_t(t_i)||_2 ||u(t_i)||_2 + \int_{t_1}^{t_2} ||u_t(s)||_2^2 ds\\ &\quad + a\int_{t_1}^{t_2}\int_\Omega |u_t(x,s)|^{m-1}|u(x,s)|dxds \qquad (4.5)\\ &\le 4C(\Omega)F(t)\sup_{t_1\le s\le t_2}||u(s)|| + C(\Omega)F^2(t)\\ &\quad + a\int_{t_1}^{t_2}\int_\Omega |u_t(x,s)|^{m-1}|u(x,s)|dxds.\end{aligned}$$

To estimate the last term of (4.5) we note that, by (3.1) and (3.4), we have

$$E(t) \ge J(t) \ge \frac{p-2}{2p}||\Delta u(t)||_2^2;$$

consequently we get, by Holder's inequality,

$$\begin{aligned}& a\int_{t_1}^{t_2}\int_\Omega |u_t(x,s)|^{m-1}|u(x,s)|dxds\\ \le\ & a\int_{t_1}^{t_2}\left(\int_\Omega |u_t(x,s)|^m dx\right)^{(m-1)/m}\left(\int_\Omega |u(x,s)|^m dx\right)^{1/m} ds \qquad (4.6)\\ \le\ & a\int_{t_1}^{t_2} ||u_t(s)||_m^{m-1}||u(s)||_m ds \le aC_*\int_{t_1}^{t_2} ||u_t(s)||_m^{m-1}||\Delta u(s)||_2 ds\\ \le\ & aC_* \sup_{t_1\le s\le t_2} E^{1/2}(s)\left(\frac{2p}{p-2}\right)^{1/2}\left(\int_{t_1}^{t_2} ||u_t(s)||_m^m ds\right)^{(m-1)/m}\\ \le\ & aC_* \sup_{t_1\le s\le t_2} E^{1/2}(s)\left(\frac{2p}{p-2}\right)^{1/2}(F(t))^{m-1}.\end{aligned}$$

A combination of (4.5), and (4.6) then yields

$$\int_{t_1}^{t_2} I(s)ds \le 4C(\Omega)F(t)\sup_{t_1\le s\le t_2}||u(s)|| + C(\Omega)F^2(t)$$

$$+aC_* \sup_{t_1 \le s \le t_2} E^{1/2}(s) \left(\frac{2p}{p-2}\right)^{1/2} (F(t))^{m-1}. \qquad (4.7)$$

$$\le C(\Omega)F^2(t) + aC_* \left(\frac{2p}{p-2}\right)^{1/2} \sup_{t_1 \le s \le t_2} E^{1/2}(s) (F(t))^{m-1}$$

$$+4C(\Omega)C_* \left(\frac{2p}{p-2}\right)^{1/2} \sup_{t_1 \le s \le t_2} E^{1/2}(s) F(t).$$

On the other hand we have, by (3.1) and (4.1),

$$\eta ||\Delta u(.,t)||_2^2 \le ||\Delta u(.,t_i)||_2^2 - b||u(.,t)||_p^p = I(t). \qquad (4.8)$$

We also have

$$\begin{aligned}
\int_{t_1}^{t_2} E(s)ds &= \frac{1}{2}\int_{t_1}^{t_2} ||u_t(s)||_2^2 ds + \int_{t_1}^{t_2} J(s)ds \\
&= \frac{1}{2}\int_{t_1}^{t_2} ||u_t(s)||_2^2 ds + \frac{1}{p}\int_{t_1}^{t_2} I(s)ds + \frac{p-2}{2p}\int_{t_1}^{t_2} ||\Delta u(s)||_2^2 ds \\
&\le \frac{C(\Omega)}{2}F^2(t) + \frac{1}{p}\int_{t_1}^{t_2} I(s)ds + \frac{p-2}{2p\eta}\int_{t_1}^{t_2} I(s)ds \qquad (4.9) \\
&\le \frac{C(\Omega)}{2}F^2(t) + \left(\frac{1}{p} + \frac{p-2}{2p\eta}\right)\int_{t_1}^{t_2} I(s)ds.
\end{aligned}$$

By combining (4.7) - (4.9) and using the fact that $E(t)$ is nonincreasing, we arrive at

$$\begin{aligned}
\int_{t_1}^{t_2} E(s)ds &\le C_2\{F^2(t) + \sup_{t \le s \le t+1} E^{1/2}(s) (F(t))^{m-1} + \sup_{t \le s \le t+1} E^{1/2}(s)F(t)\} \\
&\le C_2\{F^2(t) + E^{1/2}(t) (F(t))^{m-1} + E^{1/2}(t)F(t)\}, \qquad (4.10)
\end{aligned}$$

where C_2, here and in what follows, is a generic positive constant independent of t. Next we multiply (1.1) by u_t and integrate over $\Omega \times [t, t_2]$ to get

$$E(t) = E(t_2) + a\int_t^{t_2} ||u_t(s)||_m^m ds. \qquad (4.11)$$

We then integrate (4.11) over $[t_1, t_2]$ and use the fact $t_2 - t_1 \ge 1/2$ to obtain

$$\int_{t_1}^{t_2} E(t)ds \ge \int_{t_1}^{t_2} E(t_2)ds \ge \frac{1}{2}E(t_2);$$

hence

$$E(t_2) \le 2\int_{t_1}^{t_2} E(t)ds. \qquad (4.12)$$

By replacing in (4.11) and using (4.9) and (4.12), we get

$$\begin{aligned}
E(t) &\le 2\int_{t_1}^{t_2} E(t)ds + a\int_t^{t_2} ||u_t(s)||_m^m ds \qquad (4.13) \\
&\le C_2\{F^2(t) + E^{1/2}(t) (F(t))^{m-1} + E^{1/2}(t)F(t)\} + aF^m(t) \\
&\le C_2\{F^2(t) + E^{1/2}(t) (F(t))^{m-1} + E^{1/2}(t)F(t) + F^m(t)\}.
\end{aligned}$$

Therefore by using Young's inequality, we have

$$E(t) \leq C_2\{F^2(t) + (F(t))^{2(m-1)} + F^m(t)\}. \tag{4.14}$$

To this end we distinguish two cases :
1) if $m = 2$ then (4.14) gives

$$E(t) \leq C_2F^2(t) = C_2\{E(t) - E(t+1)\}. \tag{4.15}$$

Therefore by applying Nakao's result to (4.15) we obtain

$$E(t) \leq E(0)e^{-k[t-1]^+}, \tag{4.16}$$

for $k = \ln(C_2/(C_2 - 1))$ and $[t-1]^+ = \max\{t-1, 0\}$.
2) If $m > 2$ we then use the fact that $E(t)$ is nonincreasing and $E(t) \geq 0$ to arrive at

$$aF^m(t) = E(t) - E(t+1) \leq E(0); \tag{4.17}$$

consequently we have, from (4.14),

$$\begin{aligned} E(t) &\leq C_2\{1 + (F(t))^{2(m-2)} + F^{m-2}(t)\}F^2(t) \\ &\leq C_2\{1 + (E(0))^{2(m-2)/m} + (E(0))^{(m-2)/m}\}F^2(t) = C_2F^2(t), \end{aligned} \tag{4.18}$$

which implies that

$$E^{m/2}(t) \leq C_2F^m(t) \leq C_2\{E(t) - E(t+1)\}. \tag{4.19}$$

Again Nakao's lemma leads to

$$E(t) \leq \left(E^{(1-m/2)}(0) + \frac{1}{C_2}(\frac{m}{2} - 1)[t-1]^+\right)^{-2/(m-2)} \tag{4.20}$$

This completes the proof.
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