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Global Existence and Nonexistence in a System of Petrovsky

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In this paper we consider the nonlinearly damped semilinear Petrovsky equation

$$u_{tt} + \Delta^2 u + a u_t |u_t|^{m-2} = b u |u|^{p-2}$$

in a bounded domain, where $a, b > 0$. We prove the existence of a local weak solution and show that this solution blows up in finite time if $p > m$ and the energy is negative. We also show that the solution is global if $m \geq p$.

Key Words: nonlinear damping; nonlinear source; negative initial energy; local; global; blow up; finite time.

1. INTRODUCTION

In [4], Guesmia considered the problem

$$\begin{aligned} u_{tt}(x,t) + \Delta^2 u(x,t) + q(x)u(x,t) + g(u_t(x,t)) = 0, \quad x \in \Omega, \quad t > 0 \\ u(x,t) = \partial_\nu u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$, with a smooth boundary $\partial\Omega$, and ν is the unit outer normal on $\partial\Omega$. For g continuous, increasing, satisfying $g(0) = 0$, and q: $\Omega \to \mathbb{R}^+$, a bounded function, Guesmia [4] proved a global existence and a regularity result. He also established, under suitable growth conditions on g, decay results for weak, as well as strong, solutions. Precisely, the author showed that the solution decays exponentially if g behaves like a linear function, whereas the decay is of a polynomial order

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otherwise. Results similar to the above system, coupled with a semilinear wave equation, have been established by Guesmia [5]. Also the system composed of the equation (1.1), with $\Delta^2 u_t(x,t) + \Delta g(\Delta u(x,t))$ in the place of $q(x)u(x,t) + g(u_t(x,t))$, has been treated by Aassila and Guesmia [1], and an exponential decay theorem, through the use of an important lemma of Komornik [6], has been established.

In this paper we are concerned with the problem

$$\begin{aligned} &u_{tt} + \Delta^2 u + a u_t |u_t|^{m-2} = b u |u|^{p-2}, \quad x \in \Omega, \quad t > 0 \\ &u(x,t) = \partial_\nu u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0 \\ &u(x,0) = \phi(x), \quad u_t(x,0) = \varphi(x), \quad x \in \Omega, \end{aligned} \tag{1.2}$$

where $a, b > 0$ and $p, m > 2$. This is a problem similar to (1.1), which contains a nonlinear source term competing with the damping factor. We will establish an existence result and show that the solution continues to exist globally if $m \geq p$; however, it blows up in finite time if $m < p$. It is worth mentioning that it is only for simplicity that q is taken to be zero, $g(u_t(x,t)) = a u_t |u_t|^{m-2}$, and the source term has a power form. The same theorems could be established for more general functions.

2. LOCAL EXISTENCE

In this section, we establish a local existence result for (1.2) under suitable conditions on m and p. First we consider, for v given, the linear problem

$$\begin{aligned} &u_{tt} + \Delta^2 u + a u_t |u_t|^{m-2} = b |v|^{p-2} v, \quad x \in \Omega, \quad t > 0 \\ &u(x,t) = \partial_\nu u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0 \\ &u(x,0) = \phi(x), \quad u_t(x,0) = \varphi(x), \quad x \in \Omega, \end{aligned} \tag{2.1}$$

where u is the sought solution.

LEMMA 2.1. *Assume that*

$$\begin{aligned} &2 < p, \qquad n \leq 4 \\ &2 < p \leq 2(n-2)/n-4, \qquad n \geq 5. \end{aligned} \tag{2.2}$$

Then given any v in $C([0,T]; C_0^\infty(\Omega))$ and ϕ, φ in $C_0^\infty(\Omega)$, the problem (2.1) *has a unique solution u satisfying*

$$\begin{aligned} &u \in L^\infty((0,T); W), \quad u_{tt} \in L^\infty((0,T); L^2(\Omega)) \\ &u_t \in L^\infty((0,T); H_0^2(\Omega)) \cap L^m(\Omega \times (0,T)). \end{aligned} \tag{2.3}$$

Here $H_0^2(\Omega) = \{w \in H^2(\Omega) : w = \partial_\nu w = 0$ on $\partial\Omega\}$ and $\mathbf{W} = \{w \in H^4(\Omega) \cap H_0^2(\Omega) : \Delta w = \partial_\nu \Delta w = 0$ on $\partial\Omega\}$.

This lemma is a direct result of [7, Theorem 3.1, Chap. 1] (see also [2] and [4, Theorem 1.2]).

LEMMA 2.2. *Assume that* (2.2) *holds. Assume further that*

$$m \le 2n/(n-4), \qquad n \ge 5. \tag{2.4}$$

Then given any ϕ *in* $H_0^2(\Omega)$, φ *in* $L^2(\Omega)$, *and* v *in* $C([0,T]; H_0^2(\Omega))$, *the problem* (2.1) *has a unique weak solution*,

$$\begin{aligned} u &\in C([0,T]; H_0^2(\Omega)) \\ u_t &\in C([0,T]; L^2(\Omega)) \cap L^m(\Omega \times (0,T)). \end{aligned} \tag{2.5}$$

Moreover, we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega [u_t^2 + (\Delta u)^2](x,t)\,dx + a\int_0^t \int_\Omega |u_t(x,s)|^m\,dx\,ds \\ &\quad = \frac{1}{2}\int_\Omega [\varphi^2 + (\Delta\phi)^2](x)\,dx + b\int_0^t \int_\Omega |v|^{p-2} v u_t(x,s)\,dx\,ds, \\ &\qquad \forall\, t \in [0,T]. \end{aligned} \tag{2.6}$$

Proof. We approximate ϕ, φ by sequences $(\phi^\mu), (\varphi^\mu)$ in $C_0^\infty(\Omega)$, and v by a sequence (v^μ) in $C([0,T]; C_0^\infty(\Omega))$. We then consider the set of linear problems

$$\begin{aligned} u_{tt}^\mu + \Delta^2 u^\mu + a u_t^\mu |u_t^\mu|^{m-2} &= b|v^\mu|^{p-2} v^\mu, \quad x \in \Omega, \quad t > 0 \\ u^\mu(x,t) = \partial_\nu u^\mu(x,t) &= 0, \quad x \in \partial\Omega, \quad t > 0 \\ u^\mu(x,0) = \phi^\mu(x), \quad u_t^\mu(x,0) &= \varphi^\mu(x), \quad x \in \Omega. \end{aligned} \tag{2.7}$$

Lemma 2.1 guarantees the existence of a sequence of unique solutions (u^μ) satisfying (2.3). Now we proceed to show that the sequence (u^μ, u_t^μ) is Cauchy in

$$\mathbf{Y} := \{w : w \in C([0,T]; H_0^2(\Omega)),\ w_t \in C([0,T]; L^2(\Omega)) \cap L^m(\Omega \times (0,T))\}.$$

For this aim, we set

$$U := u^\mu - u^\nu, \qquad V := v^\mu - v^\nu.$$

It is straightforward to see that U satisfies

$$\begin{aligned} U_{tt} + \Delta^2 U + a(u_t^\mu |u_t^\mu|^{m-2} - u_t^\nu |u_t^\nu|^{m-2}) &= b(|v^\mu|^{p-2} v^\mu - |v^\nu|^{p-2} v^\nu) \\ U(x,t) &= 0, \quad x \in \partial\Omega, \quad t > 0 \\ U(x,0) = U_0(x) = \phi^\mu(x) - \phi^\nu(x), \quad U_t(x,0) &= U_1(x) = \varphi^\mu(x) - \varphi^\nu(x). \end{aligned} \tag{2.8}$$

We multiply Eq. (2.8) by U_t and integrate over $\Omega \times (0, t)$ to get

$$\frac{1}{2}\int_\Omega [U_t^2 + (\Delta U)^2](x,t)\,dx + a\int_0^t \int_\Omega (u_t^\mu |u_t^\mu|^{m-2} - u_t^\nu |u_t^\nu|^{m-2}) U_t(x,s)\,dx\,ds$$
$$= \frac{1}{2}\int_\Omega [U_1^2 + (\Delta U_0)^2](x)\,dx + b\int_0^t \int_\Omega [|v^\mu|^{p-2}v^\mu - |v^\nu|^{p-2}v^\nu]$$
$$\times U_t(x,s)\,dx\,ds. \tag{2.9}$$

We then estimate the last term in (2.9) as follows:

$$\int_\Omega \Big|[|v^\mu|^{p-2}v^\mu - |v^\nu|^{p-2}v^\nu]U_t(x,s)\Big|\,dx$$
$$\leq C\|U_t\|_2 \|V\|_{2n/(n-4)}\Big[\|v^\mu\|_{n(p-2)/2}^{p-2} + \|v^\nu\|_{n(p-2)/2}^{p-2}\Big]. \tag{2.10}$$

The Sobolev embedding and condition (2.2) give

$$\|V\|_{2n/(n-2)} \leq C\|\Delta V\|_2,$$
$$\|v^\mu\|_{n(p-2)/2}^{p-2} + \|v^\nu\|_{n(p-2)/2}^{p-2} \leq C[\|\Delta v^\mu\|_2^{p-2} + \|\Delta v^\nu\|_2^{p-2}],$$

where C is a constant depending on Ω only. Therefore (2.10) takes the form

$$\int_\Omega \Big|[|v^\mu|^{p-2}v^\mu - |v^\nu|^{p-2}v^\nu]U_t(x,s)\Big|\,dx$$
$$\leq C\|U_t\|_2 \|\Delta V\|_2 [\|\Delta v^\mu\|_2^{p-2} + \|\Delta v^\nu\|_2^{p-2}].$$

Since $(u_t^\mu |u_t^\mu|^{m-2} - u_t^\nu |u_t^\nu|^{m-2})(u_t^\mu - u_t^\nu) \geq 0$ then (2.9) yields

$$\frac{1}{2}\int_\Omega [U_t^2 + (\Delta U)^2](x,t)\,dx \leq \int_\Omega [U_1^2 + (\Delta U_0)^2](x)\,dx$$
$$+\Gamma \int_0^t \|U_t(.,s)\|_2 \|\Delta V(.,s)\|_2\,ds,$$

where Γ is a generic positive constant depending on C and the radius of the ball in $C([0,T]; H_0^2(\Omega))$ containing v^μ and v^ν. Young's inequality then gives

$$\max_{0\leq t\leq T} \int_\Omega [U_t^2 + (\Delta U)^2](x,t)\,dx \leq \Gamma \int_\Omega [U_1^2 + |\Delta U_0|^2](x)\,dx$$
$$+\Gamma T \max_{0\leq t\leq T} \int_\Omega [V_t^2 + (\Delta V)^2](x,t)\,dx.$$

Since (ϕ^μ) is Cauchy in $H_0^2(\Omega)$, (φ^μ) is Cauchy in $L^2(\Omega)$, and (v^μ) is Cauchy in $C([0,T]; H_0^2(\Omega))$, we conclude that (u^μ, u_t^μ) is Cauchy in $C([0,T];H_0^2(\Omega)) \times C([0,T];L^2(\Omega))$. To show that u_t is Cauchy in $L^m(\Omega \times (0,T))$, we use

$$\|U_t\|^m_{L^m(\Omega\times(0,T))} \le C\int_0^t \int_\Omega (u_t^\mu|u_t^\mu|^{m-2} - u_t^\nu|u_t^\nu|^{m-2})U_t(x,s)\,dx\,ds, \quad (2.11)$$

which yields, by (2.9),

$$\|U_t\|^m_{L^m(\Omega\times(0,T))} \le \Gamma\int_\Omega [U_1^2 + (\Delta U_0)^2](x)\,dx + \Gamma\int_0^T \|U_t(.,s)\|_2\|\Delta V(.,s)\|_2\,ds.$$

Therefore (u_t^μ) is Cauchy in $L^m(\Omega \times (0,T))$ and hence (u^μ) is Cauchy in **Y**. We now show that the limit u is a weak solution of (2.1) in the sense of [7]. That is for each θ in $H_0^2(\Omega)$ we must show that

$$\frac{d}{dt}\int_\Omega u_t(x,t)\theta(x)\,dx + \int_\Omega \Delta u(x,t)\Delta\theta(x)\,dx + a\int_\Omega u_t|u_t|^{m-2}(x,t)\theta(x)\,dx = b\int_\Omega |v|^{p-2}v(x,t)\theta(x)\,dx, \quad (2.12)$$

for almost all t in $[0,T]$. To establish this, we multiply Eq. (2.7) by θ and integrate over Ω, so we obtain

$$\frac{d}{dt}\int_\Omega u_t^\mu(x,t)\theta(x)\,dx + \int_\Omega \Delta u^\mu(x,t)\Delta\theta(x)\,dx + a\int_\Omega u_t^\mu|u_t^\mu|^{m-2}(x,t)\theta(x)\,dx = b\int_\Omega |v^\mu|^{p-2}v^\mu(x,t)\theta(x)\,dx. \quad (2.13)$$

As $\mu \to \infty$, we see that

$$\int_\Omega \Delta u^\mu(x,t)\Delta\theta(x)\,dx \to \int_\Omega \Delta u(x,t)\Delta\theta(x)\,dx,$$

$$\int_\Omega |v^\mu|^{p-2}v^\mu(x,t)\theta(x)\,dx \to \int_\Omega |v|^{p-2}v(x,t)\theta(x)\,dx$$

in $C([0,T])$ and

$$\int_\Omega u_t^\mu|u_t^\mu|^{m-2}(x,t)\theta(x)\,dx \to \int_\Omega u_t|u_t|^{m-2}(x,t)\theta(x)\,dx$$

in $L^1((0,T))$. We thus have $\int_\Omega u_t(x,t)\theta(x)\,dx$ $\{=\lim \int_\Omega u_t^\mu(x,t)\theta(x)\,dx\}$ is an absolutely continuous function on $[0,T]$, so (2.12) holds for almost all t in $[0,T]$. For the energy equality (2.6), we start from the energy equality for u^μ and proceed in the same way to establish it for u. To prove uniqueness, we take v^μ and v^ν and let u^μ and u^ν be the corresponding solutions of (2.1). It is clear that $U = u^\mu - u^\nu$ satisfies

$$\frac{1}{2}\int_\Omega [U_t^2 + (\Delta U)^2](x,t)\,dx + a\int_0^t\int_\Omega (u_t^\mu |u_t^\mu|^{m-2} - u_t^\nu |u_t^\nu|^{m-2})U_t(x,s)\,dx\,ds$$
$$= b\int_0^t\int_\Omega [|v^\mu|^{p-2}v^\mu - |v^\nu|^{p-2}v^\nu]U_t(x,s)\,dx\,ds. \tag{2.14}$$

If $v^\mu = v^\nu$ then (2.14) shows that $U = 0$, which implies uniqueness. This completes the proof.

Remark 2.1. Note that the condition (2.4) on m is needed so that $\int_\Omega u_t^\mu |u_t^\mu|^{m-2}(x,t)\theta(x)\,dx$ and $\int_\Omega u_t|u_t|^{m-2}(x,t)\theta(x)\,dx$ make sense.

THEOREM 2.3. *Assume that* (2.2) *and* (2.4) *hold. Then given any* ϕ *in* $H_0^2(\Omega)$, *and* φ *in* $L^2(\Omega)$, *the problem* (1.2) *has a unique weak solution* $u \in \mathbf{Y}$, *for* T *small enough.*

Proof. For $M > 0$ large and $T > 0$, we define a class of functions $Z(M,T)$ which consists of all functions w in $\mathbf{Y}$ satisfying the initial conditions of (1.2) and

$$\max_{0\le t\le T} \frac{1}{2}\int_\Omega [w_t^2 + (\Delta w)^2](x,t)\,dx + a\int_0^T\int_\Omega |w_t(x,s)|^m\,dx\,ds \le M^2. \tag{2.15}$$

$Z(M,T)$ is nonempty if M is large enough. This follows from the trace theorem (see [8]). We also define the map f from $Z(M,T)$ into $\mathbf{Y}$ by $u := f(v)$, where u is the unique solution of the linear problem (2.1). We would like to show, for M sufficiently large and T sufficiently small, that f is a contraction from $Z(M,T)$ into itself.

By using the energy equality (2.5) we get

$$\int_\Omega [u_t^2 + (\Delta u)^2](x,t)\,dx + 2a\int_0^t\int_\Omega |u_t(x,s)|^m\,dx\,ds$$
$$\le \int_\Omega [u_1^2 + (\Delta u_0)^2](x)\,dx + 2b\int_0^t\int_\Omega |v|^{p-1}|u_t|(x,s)\,dx\,ds, \quad \forall t \in [0,T]$$
$$\le \int_\Omega [u_1^2 + (\Delta u_0)^2](x)\,dx + 2b\int_0^t \|u_t\|_2 \|\Delta v\|_2^{p-1}, \quad \forall t \in [0,T]; \tag{2.16}$$

consequently

$$\|u\|_{\mathbf{Y}}^2 \le C\int_\Omega [u_1^2 + (\Delta u_0)^2](x)\,dx + CM^{p-1}T\|u\|_{\mathbf{Y}},$$

where C is independant of M. By choosing M large enough and T sufficiently small, (2.15) is satisfied; hence $u \in Z(M,T)$. This shows that f maps $Z(M,T)$ into itself.

Next we verify that f is a contraction. For this aim we set $U = u - \bar{u}$ and $V = v - \bar{v}$, where $u = f(v)$ and $\bar{u} = f(\bar{v})$. It is straightforward to see that U satisfies

$$\begin{aligned} U_{tt} + \Delta^2 U + a|u_t|^{m-2}u_t - a|\bar{u}_t|^{m-2}\bar{u}_t &= b|v|^{p-2}v - b|\bar{v}|^{p-2}\bar{v} \\ U(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0 \qquad (2.17) \\ U(x,0) = U_t(x,0) = 0, \quad x \in \Omega. \end{aligned}$$

By multiplying Eq. (2.17) by U_t and integrating over $\Omega \times (0,t)$, we arrive at

$$\begin{aligned} &\int_\Omega [U_t^2 + (\Delta U)^2](x,t)\,dx + \int_0^t \int_\Omega (|u_t|^{m-2}u_t - |\bar{u}_t|^{m-2}\bar{u}_t)U_t(x,s)\,dx\,ds \\ &\quad \le C \int_0^t \int_\Omega ||v|^{p-2}v - |\bar{v}|^{p-2}\bar{v}||U_t|(x,s)\,dx\,ds. \end{aligned} \tag{2.18}$$

By using (2.2), (2.10), and (2.11), we obtain

$$\begin{aligned} &\int_\Omega [U_t^2 + (\Delta U)^2](x,t)\,dx + \int_0^t \int_\Omega |U_t(x,s)|^m\,dx\,ds \\ &\quad \le \Gamma \int_0^t \|U_t\|_2 \|\Delta V\|_2 (\|\Delta v\|_2^{p-2} + \|\Delta \bar{v}\|_2^{p-2})(.,s)\,ds. \end{aligned}$$

Thus we have

$$\|U\|_{\mathbf{Y}}^2 \le CTM^{p-2}\|V\|_{\mathbf{Y}}^2. \tag{2.19}$$

By choosing T so small that $\Gamma TM^{p-2} < 1$, (2.19) shows that f is a contraction. The contraction mapping theorem then guarantees the existence of a unique u satisfying $u = f(u)$. Obviously it is a solution of (1.2). The uniqueness of this solution follows from the energy inequality (2.18). The proof is completed.

3. BLOW-UP RESULT

In this section we show that the solution (2.5) blows up in finite time if $p > m$ and $E(0) < 0$, where

$$E(t) := \frac{1}{2}\int_\Omega [u_t^2 + (\Delta u)^2](x,t)\,dx - \frac{b}{p}\int_\Omega |u(x,t)|^p\,dx. \tag{3.1}$$

LEMMA 3.1. *Suppose that* (2.2) *holds. Then there exists a positive constant* $C > 1$, *depending on* Ω *only, such that*

$$\|u\|_p^s \le C(\,|\Delta u\|_2^2 + \|u\|_p^p) \tag{3.2}$$

for any $u \in H_0^2(\Omega)$ *and* $2 \le s \le p$.

Proof. If $\|u\|_p \leq 1$ then $\|u\|_p^s \leq \|u\|_p^2 \leq C\|\Delta u\|_2^2$ by Sobolev embedding theorems and the boundary conditions. If $\|u\|_p > 1$ then $\|u\|_p^s \leq \|u\|_p^p$. Therefore (3.2) follows.

We set

$$H(t) := -E(t) \tag{3.3}$$

and use, throughout this section, C to denote a generic positive constant depending on Ω only. As a result of (3.1)–(3.3), we have

COROLLARY 3.2. *Let the assumptions of the lemma hold. Then we have*

$$\|u\|_p^s \leq C(|H(t)| + \|u_t\|_2^2 + \|u\|_p^p) \tag{3.4}$$

for any $u \in H_0^2(\Omega)$ *and* $2 \leq s \leq p$.

THEOREM 3.3. *Let the conditions of the Theorem* 2.3 *be fulfilled. Assume further that*

$$E(0) < 0. \tag{3.5}$$

Then the solution (2.5) *blows up in finite time.*

Proof. We multiply Eq. (1.2) by $-u_t$ and integrate over Ω to get

$$H'(t) = a\int_\Omega |u_t(x,t)|^m\,dx \geq 0,$$

for almost every t in $[0, T)$ since $H(t)$ is absolutely continuous (see [2]); hence

$$0 < H(0) \leq H(t) \leq \frac{b}{p}\|u\|_p^p, \tag{3.6}$$

for every t in $[0, T)$, by virtue of (3.1) and (3.3). We then define

$$L(t) := H^{1-\alpha}(t) + \varepsilon\int_\Omega uu_t(x,t)\,dx \tag{3.7}$$

for ε small to be chosen later and

$$0 < \alpha \leq \min\left\{\frac{(p-2)}{2p}, \frac{(p-m)}{p(m-1)}\right\}. \tag{3.8}$$

By taking a derivative of (3.7) and using Eq. (1.2) we obtain

$$\begin{aligned} L'(t) &:= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega [u_t^2 - (\Delta u)^2](x,t)\,dx \\ &\quad + \varepsilon b\int_\Omega |u(x,t)|^p\,dx - a\varepsilon\int_\Omega |u_t|^{m-2}u_t u(x,t)\,dx. \end{aligned} \tag{3.9}$$

We then exploit Young's inequality,

$$XY \le \frac{\delta^r}{r} X^r + \frac{\delta^{-q}}{q} Y^q, \quad X, Y \ge 0, \quad \delta > 0, \quad \frac{1}{r} + \frac{1}{q} = 1,$$

for $r = m$ and $q = m/(m-1)$ to estimate the last term in (3.9) as

$$\int_\Omega |u_t|^{m-1}|u|\,dx \le \frac{\delta^m}{m}\|u\|_m^m + \frac{m-1}{m}\delta^{-m/(m-1)}\|u_t\|_m^m,$$

which yields, by substitution in (3.9),

$$\begin{aligned} L'(t) \ge & \left[(1-\alpha)H^{-\alpha}(t) - \frac{m-1}{m}\varepsilon\delta^{-m/(m-1)}\right]H'(t) \\ & + \varepsilon\int_\Omega [u_t^2 - (\Delta u)^2](x,t)\,dx + \varepsilon\left[pH(t) + \frac{p}{2}\int_\Omega [u_t^2 + (\Delta u)^2](x,t)\,dx\right] \\ & - \varepsilon a\frac{\delta^m}{m}\|u\|_m^m, \quad \forall \delta > 0. \end{aligned} \tag{3.10}$$

Of course (3.10) remains valid even if δ is time dependent, since the integral is taken over the x variable. Therefore by taking δ so that $\delta^{-m/(m-1)} = kH^{-\alpha}(t)$, for large k to be specified later, and substituting in (3.10) we arrive at

$$\begin{aligned} L'(t) \ge & \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t) \\ & + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)\,dx + \varepsilon\left(\frac{p}{2}-1\right)\int_\Omega (\Delta u(x,t))^2\,dx \\ & + \varepsilon\left[pH(t) - \frac{k^{1-m}}{m}aH^{\alpha(m-1)}(t)\|u\|_m^m\right]. \end{aligned} \tag{3.11}$$

By exploiting (3.6) and the inequality $\|u\|_m^m \le C\,\|u\|_p^m$, we obtain

$$H^{\alpha(m-1)}(t)\|u\|_m^m \le \left(\frac{b}{p}\right)^{\alpha(m-1)} C\|u\|_p^{m+\alpha p(m-1)};$$

hence (3.11) yields

$$\begin{aligned} L'(t) \ge & \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t) \\ & + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)\,dx + \varepsilon\left(\frac{p}{2}-1\right)\int_\Omega (\Delta u(x,t))^2\,dx \\ & + \varepsilon\left[pH(t) - \frac{k^{1-m}}{m}a\left(\frac{b}{p}\right)^{\alpha(m-1)} C\|u\|_p^{m+\alpha p(m-1)}\right]. \end{aligned} \tag{3.12}$$

We then use Corollary 3.2 and relation (3.8), for $s = m + \alpha p(m-1) \le p$, to deduce from (3.12),

$$L'(t) \ge \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)\,dx + \varepsilon\left(\frac{p}{2}-1\right)\int_\Omega |\nabla u|^2(x,t)\,dx + \varepsilon[pH(t) - C_1 k^{1-m}\{H(t) + \|u_t\|_2^2 + \|u\|_p^p\}], \tag{3.13}$$

where $C_1 = a(b/p)^{\alpha(m-1)}C/m$. By noting that

$$H(t) = \frac{b}{p}\|u\|_p^p - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\Delta u\|_2^2$$

and writing $p = (p+2)/2 + (p-2)/2$, the estimate (3.13) gives

$$L'(t) \ge \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\frac{p-2}{4}\|\Delta u\|_2^2 + \varepsilon\left[\left(\frac{p+2}{2} - C_1k^{1-m}\right)H(t) + \left(\frac{p-2}{2p}b - C_1k^{1-m}\right)\|u\|_p^p + \left(\frac{p+6}{4} - C_1k^{1-m}\right)\|u_t\|_2^2\right]. \tag{3.14}$$

At this point, we choose k large enough so that the coefficients of $H(t)$, $\|u_t\|_2^2$, and $\|u\|_p^p$ in (3.14) are strictly positive; hence we get

$$L'(t) \ge \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\gamma[H(t) + \|u_t\|_2^2 + \|u\|_p^p], \tag{3.15}$$

where $\gamma > 0$ is the minimum of these coefficients. Once k is fixed (hence γ), we pick ε small enough so that $(1-\alpha) - \varepsilon k(m-1)/m \ge 0$ and

$$L(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0u_1(x)\,dx > 0.$$

Therefore (3.15) takes the form

$$L'(t) \ge \gamma\varepsilon[H(t) + \|u_t\|_2^2 + \|u\|_p^p]. \tag{3.16}$$

Consequently we have

$$L(t) \ge L(0) > 0, \qquad \forall\, t \ge 0.$$

Next we estimate the second term in (3.7) as follows:

$$\left|\int_\Omega uu_t(x,t)\,dx\right| \le \|u\|_2\|u_t\|_2 \le C\|u\|_p\|u_t\|_2.$$

So we have

$$\left|\int_\Omega uu_t(x,t)\,dx\right|^{1/(1-\alpha)} \le C\|u\|_p^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}.$$

Again Young's inequality gives

$$\left|\int_\Omega uu_t(x,t)\,dx\right|^{1/(1-\alpha)} \le C\Big[\|u\|_p^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\Big], \tag{3.17}$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\alpha)$ to get $\mu/(1-\alpha) = 2/(1-2\alpha) \le p$ by condition (3.8). Therefore (3.17) becomes

$$\left|\int_\Omega uu_t(x,t)\,dx\right|^{1/(1-\alpha)} \le C[\|u\|_p^s + \|u_t\|_2^2],$$

where $s = 2/(1-2\alpha) \le p$. By using Corollary 3.2 we obtain

$$\left|\int_\Omega uu_t(x,t)\,dx\right|^{1/(1-\alpha)} \le C[H(t) + \|u\|_p^p + \|u_t\|_2^2], \qquad \forall\, t \ge 0. \tag{3.18}$$

Consequently we have

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon\int_\Omega uu_t(x,t)\,dx\right)^{1/(1-\alpha)} \\ &\le 2^{1/(1-\alpha)}\left(H(t) + \left|\int_\Omega uu_t(x,t)\,dx\right|^{1/(1-\alpha)}\right) \\ &\le C(H(t) + \|u\|_p^p + \|u_t\|_2^2). \end{aligned} \tag{3.19}$$

We then combine (3.16) and (3.19), to arrive at

$$L'(t) \ge \Gamma L^{1/(1-\alpha)}(t), \tag{3.20}$$

where Γ is a constant depending on C, γ, and ε only (and hence is independent of the solution u). A simple integration of (3.20) over $(0, t)$ then yields

$$L^{\alpha/(1-\alpha)}(t) \ge \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Gamma t\alpha/(1-\alpha)}.$$

Therefore $L(t)$ blows up in a time

$$T^* \le \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}}. \tag{3.21}$$

Remark 2.1. By following the steps of the proof of Theorem 3.3 closely, one can easily see that the blow-up result holds even for $1 < m < p$. Therefore this method is a unified one for both linear and nonlinear damping cases.

Remark 2.2. The estimate (3.21) shows that $L(0)$ is larger when the blow-up takes place more quickly.

4. GLOBAL EXISTENCE

In this section, we show that the solution (2.5) is global if $m \geq p$.

THEOREM 4.1. *Assume that* (2.2) *and* (2.4) *hold such that* $m \geq p$. *Then for any* ϕ *in* $H_0^2(\Omega)$ *and* φ *in* $L^2(\Omega)$, *the problem* (1.2) *has a unique weak solution* $u \in \mathbf{Y}$, *for any* $T > 0$.

Proof. Similar to [3], we define the functional

$$F(t) := \frac{1}{2}\int_\Omega [u_t^2 + (\Delta u)^2](x, t)\, dx + \frac{b}{p}\int_\Omega |u(x, t)|^p\, dx.$$

By taking a derivative and using Eq. (1.2), we obtain

$$F'(t) = -a\|u_t\|_m^m + 2b\int_\Omega u_t u|u(x, t)|^{p-2}\, dx.$$

By using Young's inequality, we get

$$F'(t) \leq -a\|u_t\|_m^m + \delta\|u_t\|_p^p + C_\delta]\|u\|_p^p.$$

By noting that $m \geq p$, we easily see that

$$F'(t) \leq -a\|u_t\|_m^m + C\delta\|u_t\|_m^p + C_\delta\|u\|_p^p,$$

where C is a constant depending on Ω only and C_δ is a constant depending on δ. At this point we distinguish two cases: either $\|u_t\|_m > 1$, so we choose δ small enough so that $-a\|u_t\|_m^m + C\delta\|u_t\|_m^p \leq 0$, and hence $F'(t) \leq C_\delta\|u\|_p^p$. Or $\|u_t\|_m \leq 1$; in this case we have $F'(t) \leq C\delta + C_\delta\|u\|_p^p$. Therefore, in either case, we have

$$F'(t) \leq c_1 + cF(t).$$

A simple integration then yields

$$F(t) \leq (F(0) + c_1/c)e^{ct}.$$

The last estimate, together with the continuation principle, completes the proof.

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