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Nonlinear Analysis 64 (2006) 2314–2331
Nonlinear Analysis
www.elsevier.com/locate/na

Existence and decay of solutions of a viscoelastic equation with a nonlinear source

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Received 6 April 2005; accepted 15 August 2005

Abstract

In a bounded domain, we consider

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u \, d\tau = |u|^\gamma u,$$

where $\gamma > 0$, and g is a nonnegative and decaying function. We prove a local existence theorem and show, for certain initial data and suitable conditions on g and γ, that this solution is global with energy which decays exponentially or polynomially depending on the rate of the decay of the relaxation function g.

MSC: 35L05; 35L15; 35L70; 37B25

Keywords: Global existence; Exponential decay; Local existence; Nonlinear source; Polynomial decay; Relaxation function; Viscoelastic

1. Introduction

In this paper we are concerned with the following problem:

$$\begin{aligned}
&u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau) \, d\tau = |u|^\gamma u, \quad x \in \Omega, \quad t \geqslant 0, \\
&u(x,t) = 0, \\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,
\end{aligned} \tag{1}$$

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0362-546X/$ - see front matter
doi:10.1016/j.na.2005.08.015

where $\gamma > 0$ is a constant, g is a positive function satisfying some conditions to be specified later, and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$, with a smooth boundary $\partial\Omega$. This type of problems have been considered by many authors and several results concerning existence, nonexistence, and asymptotic behavior have been established. In this regard, Cavalcanti et al. [5] studied the following equation:

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau + a(x)u_t + |u|^\gamma u = 0 \quad \text{in } \Omega \times (0, \infty)$$

for $a : \Omega \to \mathbb{R}^+$, a function, which may be null on a part of the domain Ω. Under the conditions that $a(x) \geqslant a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions and

$$-\xi_1 g(t) \leqslant g'(t) \leqslant -\xi_2 g(t), \quad t \geqslant 0,$$

the authors established an exponential rate of decay. Berrimi and Messaoudi [2] improved Cavalcanti's result by introducing a different functional, which allowed them to weaken the conditions on both a and g. In particular, the function a can vanish on the whole domain Ω and consequently the geometry condition has disappeared. In [6], Cavalcanti and Oquendo considered

$$u_{tt} - k_0\Delta u + \int_0^t div[a(x)g(t-\tau)\nabla u(\tau)]\,\mathrm{d}\tau + b(x)h(u_t) + f(u) = 0,$$

under similar conditions on the relaxation function g and $a(x) + b(x) \geqslant \rho > 0$, for all $x \in \Omega$. They improved the result of [5] by establishing exponential stability for g decaying exponentially and h linear and polynomial stability for g decaying polynomially and h nonlinear. Their proof, based on the use of piecewise multipliers, is similar to the one in [5]. Though both results in [2,6] improve the earlier one in [5], the approaches as well as the functionals used, are different. Other problems, where the damping induced by the integral terms cooperating with another one acting on a part of the boundary or with a strong damping, were also discussed by Cavalcanti et al. [3,4]. A related result is the work of Kawashima [12], in which he considered a one-dimensional model equation for viscoelastic materials of integral type. The memory function is allowed to have an integrable singularity. Also, for small initial data, Munoz Rivera and Baretto [18] proved that the first- and the second-order energies of the solution to a viscoelastic plate decay exponentially provided that the kernel of the memory decays exponentially. Kirane and Tatar [13] considered a mildly damped wave equation and proved that any small internal dissipation is sufficient to uniformly stabilize the solution by means of a nonlinear feedback of memory type acting on a part of the boundary. This result was established without any restriction on the space dimension or geometrical conditions on the domain or its boundary.

For the wave equation ($g = 0$), many result concerning stabilization of solutions either by internal or boundary damping have been proved. In this regard, we mention the work of Nakao [20], in which he considered

$$u_{tt} - \Delta u + a|u_t|^{m-2}u_t + bu|u|^{p-2} = 0, \quad x \in \Omega, \ t > 0$$

and proved the existence of a unique global weak solution if $0 \leqslant p - 2 \leqslant 2/(n-2)$ if $n \geqslant 3$ (in case $n = 1$ or 2 the only requirement is $p \geqslant 2$). In addition to global existence the issue of the decay rate was also addressed. In both cases it has been shown that the energy of the solution decays algebraically if $m > 2$ and decays exponentially if $m = 2$. This improves an earlier result by Nakao [19], where the author studied the problem in an abstract setting and established a theorem concerning the decay of the solution energy only for the case $m - 2 \leqslant 2/(n-2)$ if $n \geqslant 3$. Messaoudi

[17] treated a problem related to the above equation, with a damping of the form $a(1+|u_t|^{m-2})u_t$, and showed that the solution energy decays exponentially. A similar result was also established by Benaissa and Messaoudi [1] for the case where a nonlinear source term is competing with the nonlinear damping. Also Nakao and Ono [21] extended this result to the Cauchy problem

$$u_{tt} - \Delta u + \lambda^2(x)u + g(u_t) + f(u) = 0; \quad x \in \mathbb{R}^n, \ \ t > 0,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \ \ x \in \mathbb{R}^n, \tag{2}$$

where $g(u_t)$ behaves like $|u_t|^\beta u_t$ and $f(u)$ behaves like $-bu|u|^\alpha$. Later, Ono [22] studied the global existence and the decay properties of smooth solutions to problem (2), for $f \equiv 0$ and gave sharper decay estimates of the solution without any restrictions on the data size. We refer the reader to [8–11,14] and the references therein for more stabilization results.

In this article, we consider (1) and establish a local existence result. In addition, we show that, for suitable initial data, the solution is bounded and global and that the damping caused by the integral term is enough to obtain a uniform decay of solutions. In fact our result is obtained under weaker conditions on the relaxation function g than those in [6] (see Remark 5.1). Our choice of the 'Lyapunov' functional, which was first introduced in [2], made our proof straightforward and easier than the one in [5,6]. The paper is organized as follows. In Section 2, we present some notations and material needed for our work. In Section 3, we establish the local existence. The global existence for solutions with positive and small energy is given in Section 4 and the decay result is given in Section 5.

2. Preliminaries

In this section, we present some material needed in the proof of our result. For the relaxation function g we assume

(Gl) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)\,\mathrm{d}s = l > 0.$$

(G2) There exists a positive constant ξ such that

$$g'(t) \leqslant -\xi g^p(t), \quad t \geqslant 0, \ \ 1 \leqslant p < \tfrac{3}{2}.$$

Remark 2.1. An example of functions satisfying (Gl) and (G2), respectively, are

$$g(s) = \mathrm{e}^{-as}, \quad a > 1,$$
$$g(s) = b(1+s)^{-1/(p-1)}, \quad p > 1, \ \ b < (2-p)/(p-1).$$

We will also be using the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for

$$2 \leqslant q \leqslant 2n/(n-2), \quad n \geqslant 3,$$
$$q \geqslant 2, \quad n = 1, 2 \tag{3}$$

and $L^r(\Omega) \hookrightarrow L^q(\Omega)$, for $q < r$. We will use the same embedding constant denoted by C_q; i.e

$$\|u\|_q \leqslant C_q \|\nabla u\|_2, \quad \|u\|_q \leqslant C_q \|u\|_r. \tag{4}$$

Remark 2.2. Condition (3) is needed so that the nonlinearity is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$. Condition (Gl) is necessary to guarantee the hyperbolicity of system (1).

Remark 2.3. Condition $p < \frac{3}{2}$ is made to ensure that $\int_0^\infty g^{2-p}(s)\,\mathrm{d}s < \infty$.

We introduce the following functionals:

$$J(t) = \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2},$$

$$E(t) = E(u(t), u_t(t)) = J(t) + \tfrac{1}{2}\|u_t(t)\|_2^2,$$

$$I(t) = I(u(t)) = \left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) - \|u(t)\|_{\gamma+2}^{\gamma+2}, \tag{5}$$

where

$$(g \circ v)(t) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|_2^2\,\mathrm{d}\tau. \tag{6}$$

3. Local existence result

In this section we study the local existence of solutions for system (1). For this purpose, we consider, first, a related linear problem. Then, we use the well-known contraction mapping theorem to prove the existence of solutions to the nonlinear problem. Let us now consider, for v given, the linear problem

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau + a(x)u_t = |v|^\gamma v,$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geqslant 0,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{7}$$

where u is the sought solution.

Lemma 3.1. *If $v \in C([0,T]; C_0^\infty(\Omega))$ and $u_0, u_1 \in C_0^\infty(\Omega)$, then problem (7) has a unique solution u satisfying*

$$u \in L^\infty((0,T); H_0^1(\Omega) \cap H^2(\Omega))$$

$$u_t \in L^\infty((0,T); H_0^1(\Omega)), \quad u_{tt} \in L^\infty((0,T); L^2(\Omega)). \tag{8}$$

This lemma is a direct result of [16, Theorem 3.1, Chapter 1]. It can also be established by using the Galerkin method as in [6].

The next lemma shows the existence of solutions to problem (7) for initial data with less regularity and gives an a priori estimate, which will be used for the study of the nonlinear problem.

Lemma 3.2. *Assume that*

$$0<\gamma<\frac{2}{n-2}, \quad n>2,$$
$$0<\gamma, \quad n=1,2 \tag{9}$$

holds, then given any u_0 in $H_0^1(\Omega)$, u_1 in $L^2(\Omega)$, and v in $C([0,T];H_0^1(\Omega))$, problem (7) *has a unique weak solution,*

$$u\in C([0,T];H_0^1(\Omega)), \quad u_t\in C((0,T);L^2(\Omega)). \tag{10}$$

Moreover, we have

$$\frac{1}{2}\int_\Omega\left[u_t^2+\left(1-\int_0^t g(s)\,\mathrm{d}s\right)|\nabla u|^2\right](x,t)\,\mathrm{d}x+(g\circ\nabla u)(t)$$
$$\leqslant\frac{1}{2}\int_\Omega[u_1^2+|\nabla u_0|^2](x)\,\mathrm{d}x+\int_0^t\int_\Omega|v|^\gamma vu_t(x,s)\,\mathrm{d}x\,\mathrm{d}s, \quad \forall t\in[0,T]. \tag{11}$$

Proof. We approximate u_0, u_1 by sequences (u_μ^0), (u_μ^1) in $C_0^\infty(\Omega)$, and v by a sequence (v_μ) in $C([0,T];C_0^\infty(\Omega))$. We then consider the set of linear problems

$$u_\mu''-\Delta u_\mu+\int_0^t g(t-\tau)\Delta u_\mu(\tau)\,\mathrm{d}\tau=|v_\mu|^\gamma v_\mu,$$
$$u_\mu(x,t)=0, \quad x\in\partial\Omega, \ t\geqslant 0,$$
$$u_\mu(x,0)=u_\mu^0(x), \quad u_\mu'(x,0)=u_\mu^1(x), \ x\in\Omega, \tag{12}$$

where $'$ denotes a derivative with respect to t.

Lemma 3.1 guarantees the existence of a sequence of unique solutions (u_μ) satisfying (8). Now we proceed to show that the sequence (u_μ) is Cauchy in

$$\mathbf{W}:=C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega)), \tag{13}$$

equipped with the norm

$$\|w\|_{\mathbf{W}}^2:=\max_{0\leqslant t\leqslant T}\left\{\int_\Omega[w_t^2+l|\nabla w|^2](x,t)\,\mathrm{d}x\right\}. \tag{14}$$

For this aim, we set

$$U=u_\mu-u_\nu, \quad V=v_\mu-v_\nu.$$

It is straightforward to see that U satisfies

$$U_{tt}-\Delta U+\int_0^t g(t-s)\Delta U(s)\,\mathrm{d}s=|v_\mu|^\gamma v_\mu-|v_\nu|^\gamma v_\nu, \quad x\in\Omega, \ t>0,$$
$$U(x,t)=0, \quad x\in\partial\Omega, \ t\geqslant 0,$$
$$U(x,0)=U_0(x)=u_\mu^0(x)-u_\nu^0(x), \quad x\in\Omega,$$
$$U_t(x,0)=U_1(x)=u_\mu^1(x)-u_\nu^1(x), \quad x\in\Omega. \tag{15}$$

We multiply the differential equation in (15) by U_t and integrate over $\Omega \times (0, t)$ to get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left[U_t^2 + \left(1 - \int_0^t g(s)\,\mathrm{d}s\right)|\nabla U|^2\right](x,t)\,\mathrm{d}x \\
&\quad - \frac{1}{2}\int_0^t (g' \circ \nabla U)(s)\,\mathrm{d}s + (g \circ \nabla U)(t) + \int_0^t \int_\Omega g(s)|\nabla U(s)|^2\,\mathrm{d}s \\
&\quad = \frac{1}{2}\int_\Omega [U_1^2 + |\nabla U_0|^2](x)\,\mathrm{d}x + \int_0^t \int_\Omega [|v_\mu|^\gamma v_\mu - |v_\nu|^\gamma v_\nu] U_t(x,s)\,\mathrm{d}x\,\mathrm{d}s. \qquad (16)
\end{aligned}$$

We then estimate the last term in (16) as follows:

$$\left|\int_\Omega [|v^\mu|^\gamma v^\mu - |v^\nu|^\gamma v^\nu] U_t(x,s)\,\mathrm{d}x\right| \leqslant C \int_\Omega \sup(|v^\mu|^\gamma, |v^\nu|^\gamma)|V||U_t|(x,s)\,\mathrm{d}x.$$

Using Holder's inequality with $(1/q) + (1/n) + (1/2) = 1$,

$$\left|\int_\Omega [|v_\mu|^\gamma v_\mu - |v_\nu|^\gamma v_\nu] U_t(x,s)\,\mathrm{d}x\right| \leqslant C(\|v_\mu\|_{\gamma n}^\gamma + \|v_\nu\|_{\gamma n}^\gamma)\|V\|_{2n/(n-2)}\|U_t\|_2. \qquad (17)$$

The Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$ gives

$$\|V\|_{2n/(n-2)} \leqslant C\|\nabla V\|_2, \quad \|v_\mu\|_{\gamma n}^\gamma + \|v_\nu\|_{\gamma n}^\gamma \leqslant C(\|\nabla v_\mu\|_2^\gamma + \|\nabla v_\nu\|_2^\gamma),$$

where C is a constant depending on Ω, l, and γ only. Therefore (17) takes the form

$$\left|\int_\Omega [|v_\mu|^\gamma v_\mu - |v_\nu|^\gamma v_\nu] U_t(x,s)\,\mathrm{d}x\right| \leqslant C\|U_t\|_2\|\nabla V\|_2(\|\nabla v_\mu\|_2^\gamma + \|\nabla v_\nu\|_2^\gamma). \qquad (18)$$

By using (Gl), (18), and the fact that

$$-\frac{1}{2}\int_0^t (g' \circ \nabla U)(s)\,\mathrm{d}s + (g \circ \nabla U)(t) + \int_0^t \int_\Omega g(s)|\nabla U(s)|^2\,\mathrm{d}s \geqslant 0,$$

estimate (16) yields

$$\begin{aligned}
&\frac{1}{2}\int_\Omega [U_t^2 + l|\nabla U|^2](x,t)\,\mathrm{d}x \\
&\quad \leqslant \frac{1}{2}\int_\Omega \left[U_t^2 + \left(1 - \int_0^t g(s)\,\mathrm{d}s\right)|\nabla U|^2\right](x,t)\,\mathrm{d}x \\
&\quad \leqslant \frac{1}{2}\int_\Omega [U_1^2 + |\nabla U_0|^2](x)\,\mathrm{d}x + \Gamma \int_0^t \|U_t(.,s)\|_2\|\nabla V(.,s)\|_2\,\mathrm{d}s,
\end{aligned}$$

where Γ is a positive constant depending only on Ω, l, γ, and the radius of the ball $B_R(0) \subset C([0,T]; H_0^1(\Omega))$ centered at the origin and containing the sequences (v_μ) and (v_ν). Gronwall's and Young's inequalities yield

$$\|U\|_{\mathbf{W}}^2 \leqslant \Gamma \int_\Omega [U_1^2 + |\nabla U_0|^2](x)\,\mathrm{d}x + \Gamma T\|V\|_{\mathbf{W}}^2. \qquad (19)$$

Since (u_μ^0) is Cauchy in $H_0^1(\Omega)$, (u_μ^1) is Cauchy in $L^2(\Omega)$, and (v_μ) is Cauchy in $C([0,T]; H_0^1(\Omega))$, we conclude that (u_μ) is Cauchy in $\mathbf{W}$; hence (u_μ) converges to a limit u in $\mathbf{W}$. We now show that this limit is a weak solution. That is for each θ in $H_0^1(\Omega)$, we must show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t(x,t)\theta(x)\,\mathrm{d}x + \int_\Omega \nabla u(x,t).\nabla\theta(x)\,\mathrm{d}x \\ - \int_0^t \int_\Omega g(t-\tau)\nabla u(x,\tau).\nabla\theta(x)\,\mathrm{d}\tau\,\mathrm{d}x = \int_\Omega |v|^\gamma v(x,t)\theta(x)\,\mathrm{d}x, \quad \forall t\in[0,T]. \tag{20}$$

To establish this, we multiply the differential equation in (12) by θ and integrate over Ω, so we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u'_\mu(x,t)\theta(x)\,\mathrm{d}x + \int_\Omega \nabla u_\mu(x,t).\nabla\theta(x)\,\mathrm{d}x \\ - \int_0^t \int_\Omega g(t-\tau)\nabla u_\mu(x,\tau).\nabla\theta(x)\,\mathrm{d}\tau\,\mathrm{d}x = \int_\Omega |v_\mu|^\gamma v_\mu(x,t)\theta(x)\,\mathrm{d}x. \tag{21}$$

As $\mu \longrightarrow \infty$, we have

$$\int_\Omega \nabla u_\mu(x,t).\nabla\theta(x)\,\mathrm{d}x \longrightarrow \int_\Omega \nabla u(x,t).\nabla\theta(x)\,\mathrm{d}x, \\ \int_0^t\int_\Omega g(t-\tau)\nabla u_\mu(x,\tau).\nabla\theta(x)\,\mathrm{d}\tau\,\mathrm{d}x \longrightarrow \int_0^t\int_\Omega g(t-\tau)\nabla u(x,\tau).\nabla\theta(x)\,\mathrm{d}\tau\,\mathrm{d}x, \\ \int_\Omega |v_\mu|^\gamma(x,t)v_\mu(x,t)\theta(x)\,\mathrm{d}x \longrightarrow \int_\Omega |v|^\gamma(x,t)v(x,t)\theta(x)\,\mathrm{d}x, \tag{22}$$

in $C([0,T])$ since u_μ converges strongly (hence weakly) to u in $\mathbf{W}$ (see [15,7]), so (20) holds for all t in $[0,T]$. For the energy inequality (11), we start from the energy equality for u_μ and proceed in the same way to establish it for u. The uniqueness follows from (11). This completes the proof. □

Now, we are ready to state and prove our main local existence result.

Theorem 3.3. *Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and suppose that* (9) *holds. Then problem* (1) *has a unique weak solution $u \in \mathbf{W}$ for T small enough.*

Proof. For $M > 0$ large and $T > 0$, we define a class of functions $\mathbf{Z}(M,T)$ which consists of all functions w in $\mathbf{W}$ satisfying the initial conditions of (1) and

$$\|w\|_{\mathbf{W}} \leqslant M. \tag{23}$$

$\mathbf{Z}(M,T)$ is nonempty if M is large enough. This follows from the trace theorem [8]. We also define the map f from $\mathbf{Z}(M,T)$ into $\mathbf{W}$ by $u := f(v)$, where u is the unique solution of the linear problem (7). We would like to show that f is a contraction from $\mathbf{Z}(M,T)$ into itself. For this, we make use of the energy inequality (11) which yields

$$\int_\Omega \left[u_t^2 + \left(1-\int_0^t g(s)\,\mathrm{d}s\right)|\nabla u|^2\right](x,t)\,\mathrm{d}x \\ \leqslant \int_\Omega [u_1^2 + |\nabla u_0|^2](x)\,\mathrm{d}x + 2\int_0^t\int_\Omega |v|^\gamma v.u_t(x,s)\,\mathrm{d}x\,\mathrm{d}s \\ \leqslant \int_\Omega [u_1^2 + |\nabla u_0|^2](x)\,\mathrm{d}x + 2\int_0^t \|\nabla v\|_2^{\gamma+1}\|u_t\|_2\,\mathrm{d}s, \quad \forall t\in[0,T].$$

This leads to

$$\|u\|_{\mathbf{W}}^2 \leqslant C \int_\Omega [u_1^2 + |\nabla u_0|^2](x)\,\mathrm{d}x + CM^{\gamma+1}T\|u\|_{\mathbf{W}},$$

where C is independent of M. By using Holder's inequality, we arrive at

$$\|u\|_{\mathbf{W}}^2 \leqslant C \int_\Omega [u_1^2 + |\nabla u_0|^2](x)\,\mathrm{d}x + M^{\gamma+1}T\left[\frac{M^{\gamma+1}T}{2}C^2 + \frac{1}{2TM^{\gamma+1}}\|u\|_{\mathbf{W}}^2\right].$$

Hence, we obtain

$$\|u\|_{\mathbf{W}}^2 \leqslant 2C \int_\Omega [u_1^2 + |\nabla u_0|^2](x)\,\mathrm{d}x + M^{2(\gamma+1)}T^2C^2. \tag{24}$$

By choosing M large enough so that $C\int_\Omega [u_1^2 + |\nabla u_0|^2](x)\,\mathrm{d}x \leqslant \frac{1}{2}M^2$ then T sufficiently small so that $M^{2(\gamma+1)}T^2C^2 \leqslant \frac{1}{2}M^2$, (23) is satisfied; hence $u \in \mathbf{Z}(M,T)$. This shows that f maps $\mathbf{Z}(M,T)$ into itself.

Next, we verify that f is a contraction. To this end we set $U = u - u^-$ and $V = v - v^-$, where $u = f(v)$ and $u^- = f(v^-)$. It is straightforward to verify that U satisfies

$$\begin{aligned} &U_{tt} - \Delta U + \int_0^t g(t-s)\Delta U(s)\,\mathrm{d}s = |v|^\gamma v - |v^-|^\gamma v^-, \quad x \in \Omega, \quad t > 0, \\ &U(x,t) = 0, \quad x \in \partial\Omega, \quad t \geqslant 0, \\ &U(x,0) = U_t(x,0) = 0, \quad x \in \Omega. \end{aligned} \tag{25}$$

By multiplying the differential equation in (25) by U_t and integrating over $\Omega \times (0,t)$, we arrive at

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left[U_t^2 + \left(1 - \int_0^t g(s)\,\mathrm{d}s\right)|\nabla U|^2\right](x,s)\,\mathrm{d}s\,\mathrm{d}x - \frac{1}{2}\int_0^t (g' \circ \nabla U)(s)\,\mathrm{d}s \\ &\quad + (g \circ \nabla U)(t) + \int_\Omega \int_0^t g(s)|\nabla U(s)|^2\,\mathrm{d}s \\ &\qquad = \int_\Omega \int_0^t [|v|^\gamma v - |v^-|^\gamma v^-]U_t(x,s)\,\mathrm{d}s\,\mathrm{d}x. \end{aligned} \tag{26}$$

By using (Gl), (9), and (18), we obtain

$$\begin{aligned} &\int_\Omega [U_t^2 + l|\nabla U|^2](x,t)\,\mathrm{d}x \\ &\quad \leqslant \Gamma \int_0^t \|U_t\|_2 \|\nabla V\|_2 (\|\nabla v\|_2^\gamma + \|\nabla v^-\|_2^\gamma)(s)\,\mathrm{d}s. \end{aligned} \tag{27}$$

Thus we have

$$\|U\|_{\mathbf{W}} \leqslant \Gamma T M^\gamma \|V\|_{\mathbf{W}}. \tag{28}$$

By choosing T so small that $\Gamma T M^\gamma < 1$, (28) shows that f is a contraction. The contraction mapping theorem then guarantees the existence of a unique u satisfying $u = f(u)$. Obviously it is a solution of (1). The uniqueness of this solution follows from the energy inequality (11). The proof is completed. □

4. Global existence

Before we state and prove our global existence result, we need the following:

Lemma 4.1. *Suppose that* (Gl), (G2), (9) *hold, and* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *If* u *is the solution of* (1) *then the "modified" energy functional satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 \leqslant \frac{1}{2}(g' \circ \nabla u)(t) \leqslant 0, \tag{29}$$

for almost every $t \in [0, T]$.

Proof. By multiplying the differential equation in (1) by u_t and integrating over Ω, using integration by parts and hypotheses (Gl) and (G2) we obtain (29) for any regular solution. This remains valid for weak solutions by simple density argument.

See [5,2] for detailed computations. $\square$

Lemma 4.2. *Suppose that* (Gl), (G2) *and* (9) *hold, if* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *such that*

$$\beta = \frac{C_*^{\gamma+2}}{l} \left(\frac{2(\gamma+2)}{\gamma l} E(u_0, u_1) \right)^{\gamma/2} < 1$$

$$I(u_0) > 0, \tag{30}$$

where C_* *is the best Poincaré constant, then* $I(u(t)) > 0$, $\forall t > 0$.

Proof. Since $I(u_0) > 0$, then there exists (by continuity) $T_m < T$ such that

$$I(u(t)) \geqslant 0, \quad \forall t \in [0, T_m],$$

this gives

$$\begin{aligned} J(t) &= \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right) \|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2} \\ &= \left(\frac{\gamma}{2(\gamma+2)}\right)\left(\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right) + \frac{1}{\gamma+2} I(t) \\ &\geqslant \left(\frac{\gamma}{2(\gamma+2)}\right)\left(\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right). \end{aligned} \tag{31}$$

By using (Gl), (5), (29), and (31) we easily have

$$\begin{aligned} l\|\nabla u(t)\|_2^2 &\leqslant \left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 \leqslant \left(\frac{2(\gamma+2)}{\gamma}\right) J(t) \\ &\leqslant \left(\frac{2(\gamma+2)}{\gamma}\right) E(t) \leqslant \left(\frac{2(\gamma+2)}{\gamma}\right) E(u_0, u_1), \quad \forall t \in [0, T_m]. \end{aligned} \tag{32}$$

We then exploit (Gl), (4), (30), and (32) to obtain

$$\begin{aligned}\|u(t)\|_{\gamma+2}^{\gamma+2} &\leqslant C_*^{\gamma+2}\|\nabla u(t)\|_2^{\gamma+2} \\ &\leqslant \frac{C_*^{\gamma+2}}{l}\|\nabla u(t)\|_2^{\gamma} l\|\nabla u(t)\|_2^2 \leqslant \beta l\|\nabla u(t)\|_2^2 \\ &\leqslant \beta\left(1-\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 \\ &< \left(1-\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2, \quad \forall t \in [0, T_m].\end{aligned} \tag{33}$$

Therefore,

$$I(t) = \left(1-\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) - \|u(t)\|_{\gamma+2}^{\gamma+2} > 0,$$

for all $t \in [0, T_m]$. By repeating this procedure, and using the fact that

$$\lim_{t\to T_m} \frac{C_*^{\gamma+2}}{l}\left(\frac{2(\gamma+2)}{\gamma l}E(u_0, u_1)\right)^{\gamma/2} \leqslant \beta < 1,$$

T_m is extended to T. □

Theorem 4.3. *Suppose that* (Gl), (G2) *and* (9) *hold. If* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and satisfies* (30). *Then the solution is global and bounded.*

Proof. It suffices to show that

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \tag{34}$$

is bounded independently of t. To achieve this, we use (5), (29), and (31) to get

$$\begin{aligned}E(0) \geqslant E(t) &= J(t) + \frac{1}{2}\|u_t(t)\|_2^2 \\ &\geqslant \frac{\gamma}{2(\gamma+2)}[l\|\nabla u(x,t)\|_2^2 + (g \circ \nabla u)(x,t)] + \frac{1}{2}\|u_t(x,t)\|_2^2 + \frac{1}{\gamma+2}I(t) \\ &\geqslant \frac{\gamma}{2(\gamma+2)}l\|\nabla u(x,t)\|_2^2 + \frac{1}{2}\|u_t(x,t)\|_2^2,\end{aligned} \tag{35}$$

since $I(t)$ and $(g \circ \nabla u)(t)$ are positive. Therefore

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leqslant CE(0),$$

where C is a positive constant, which depends only on γ and l. □

5. Decay of solutions

In this section we state and prove our decay result. For this purpose, we use the functional of [2], namely,

$$F(t) := E(t) + \varepsilon_1\Phi(t) + \varepsilon_2\Psi(t), \tag{36}$$

where ε_1 and ε_2 are positive constants and

$$\Phi(t) := \int_\Omega u u_t \,\mathrm{d}x \tag{37}$$

$$\Psi(t) := -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))\,\mathrm{d}\tau\,\mathrm{d}x.$$

It is easy to check that, by using (4), (29)–(32), and for ε_1 and ε_2 small enough,

$$\alpha_1 F(t) \leqslant E(t) \leqslant \alpha_2 F(t) \tag{38}$$

holds for two positive constants α_1 and α_2.

Lemma 5.1. *Assume that* (Gl), (G2), *and* (9) *hold. Then the functional*

$$\Phi(t) := \int_\Omega u u_t \,\mathrm{d}x$$

satisfies, *along the solution of* (1),

$$\Phi'(t) \leqslant \int_\Omega u_t^2 \,\mathrm{d}x - \frac{l}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{l}\left[\int_0^t g^{2-p}(\tau)\,\mathrm{d}\tau\right](g^p \circ \nabla u)(t). + \|u\|_{\gamma+2}^{\gamma+2}. \tag{39}$$

Proof. By using the differential equation in (1), we easily see that

$$\begin{aligned}\Phi'(t) &= \int_\Omega (u u_{tt} + u_t^2)\,\mathrm{d}x = \int_\Omega u_t^2\,\mathrm{d}x - \int_\Omega |\nabla u(t)|^2\,\mathrm{d}x\\ &\quad + \int_\Omega \nabla u(t) \int_0^t g(t-\tau)\nabla u(\tau)\,\mathrm{d}\tau\,\mathrm{d}x + \int_\Omega |u(t)|^{\gamma+2}\,\mathrm{d}x.\end{aligned} \tag{40}$$

We now estimate the third term in the right side of (40) as follows:

$$\begin{aligned}&\int_\Omega \nabla u(t).\int_0^t g(t-\tau)\nabla u(\tau)\,\mathrm{d}\tau\,\mathrm{d}x\\ &\quad\leqslant \frac{1}{2}\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-\tau)|\nabla u(\tau)|\,\mathrm{d}\tau\right)^2\mathrm{d}x\\ &\quad\leqslant \frac{1}{2}\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau)-\nabla u(t)| + |\nabla u(t)|)\,\mathrm{d}\tau\right)^2\mathrm{d}x.\end{aligned} \tag{41}$$

We then use Young's inequality to obtain, for any $\eta > 0$,

$$\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|)\,\mathrm{d}\tau\right)^2 \mathrm{d}x \\
&\quad \leqslant \int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)|)\,\mathrm{d}\tau\right)^2 \mathrm{d}x + \int_\Omega \left(\int_0^t g(t-\tau)|\nabla u(t)|\,\mathrm{d}\tau\right)^2 \mathrm{d}x \\
&\qquad + 2\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)|)\,\mathrm{d}\tau\right)\left(\int_0^t g(t-\tau)|\nabla u(t)|\,\mathrm{d}\tau\right) \mathrm{d}x \\
&\quad \leqslant (1+\eta)\int_\Omega \left(\int_0^t g(t-\tau)|\nabla u(t)|\,\mathrm{d}\tau\right)^2 \mathrm{d}x \\
&\qquad + \left(1+\frac{1}{\eta}\right)\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)|)\,\mathrm{d}\tau\right)^2 \mathrm{d}x. \qquad (42)
\end{aligned}$$

Direct calculations (see also [6, Lemma 1.1]) yield

$$\begin{aligned}
&\left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)|)\,\mathrm{d}\tau\right)^2 \mathrm{d}x \\
&\quad \leqslant \int_0^t g^{2-p}(\tau)\,\mathrm{d}\tau \int_0^t g^p(t-\tau)|\nabla u(\tau) - \nabla u(t)|^2\,\mathrm{d}\tau\,\mathrm{d}x.
\end{aligned}$$

Thus, by using the fact $\int_0^t g(s)\,\mathrm{d}s \leqslant \int_0^\infty g(s)\,\mathrm{d}s = 1 - l$, (42) becomes

$$\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|)\,\mathrm{d}\tau\right)^2 \mathrm{d}x \\
&\quad \leqslant (1+\eta)(1-l)^2 \int_\Omega |\nabla u(t)|^2\,\mathrm{d}x \\
&\qquad + \left(1+\frac{1}{\eta}\right)\int_0^t g^{2-p}(\tau)\,\mathrm{d}\tau \int_\Omega \int_0^t g^p(t-\tau)|\nabla u(\tau) - \nabla u(t)|^2\,\mathrm{d}\tau\,\mathrm{d}x. \qquad (43)
\end{aligned}$$

By combining (40)–(43), we arrive at

$$\begin{aligned}
\Phi'(t) &\leqslant \int_\Omega u_t^2\,\mathrm{d}x - \int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega |u|^{\gamma+2}\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x \\
&\quad + \frac{1}{2}(1+\eta)(1-l)^2 \int_\Omega |\nabla u(t)|^2\,\mathrm{d}x \\
&\quad + \left(1+\frac{1}{\eta}\right)\int_0^t g^{2-p}(\tau)\,\mathrm{d}\tau \int_\Omega \int_0^t g^p(t-\tau)|\nabla u(\tau) - \nabla u(t)|^2\,\mathrm{d}\tau\,\mathrm{d}x \\
&\leqslant \int_\Omega u_t^2\,\mathrm{d}x - \int_\Omega |u|^{\gamma+2}\,\mathrm{d}x + \frac{1}{2}[-1 + (1+\eta)(1-l)^2]\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x \\
&\quad + \left(1+\frac{1}{\eta}\right)\int_0^t g^{2-p}(\tau)\,\mathrm{d}\tau \int_\Omega \int_0^t g^p(t-\tau)|\nabla u(\tau) - \nabla u(t)|^2\,\mathrm{d}\tau\,\mathrm{d}x. \qquad (44)
\end{aligned}$$

By choosing $\eta = l/(1-l)$, (39) is established. $\square$

Lemma 5.2. *Assume that* (G1), (G2), *and* (9) *hold. Then the functional*

$$\Psi(t) := -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))\,\mathrm{d}\tau\,\mathrm{d}x$$

satisfies, *along the solution of* (1),

$$\begin{aligned}\Psi'(t) &\leqslant \delta\left\{1+2(1-l)^2 + C_p^{2(\gamma+1)}\left(\frac{2(\gamma+2)E(0)}{\gamma l}\right)^\gamma\right\}\|\nabla u(t)\|_2^2 \\ &\quad + \left\{2\delta + \frac{1}{4\delta}(1+C_p)\right\}\left[\int_0^t g^{2-p}(\tau)\,\mathrm{d}\tau\right](g^p\circ\nabla u)(t) \\ &\quad + \frac{g(0)}{4\delta}C_p((-g'\circ\nabla u)(t)) + \left\{\delta - \int_0^t g(s)\,\mathrm{d}s\right\}\int_\Omega u_t^2\,\mathrm{d}x, \end{aligned} \tag{45}$$

for any $\delta > 0$.

Proof. Direct calculations give

$$\begin{aligned}\Psi'(t) &= -\int_\Omega u_{tt}\int_0^t g(t-\tau)(u(t)-u(\tau))\,\mathrm{d}\tau\,\mathrm{d}x \\ &\quad - \int_\Omega u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))\,\mathrm{d}\tau\,\mathrm{d}x - \left(\int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega u_t^2\,\mathrm{d}x \\ &= \int_\Omega \nabla u(t).\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}x \\ &\quad - \int_\Omega\left(\int_0^t g(t-\tau)(\nabla u(\tau)-\nabla u(t))\,\mathrm{d}\tau\right).\left(\int_0^t g(t-\tau)\nabla u(\tau)\,\mathrm{d}\tau\right)\mathrm{d}x \\ &\quad - \int_\Omega |u|^\gamma u\left(\int_0^t g(t-\tau)(u(t)-u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}x \\ &\quad - \int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))\,\mathrm{d}\tau\,\mathrm{d}x - \left(\int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega u_t^2\,\mathrm{d}x. \end{aligned} \tag{46}$$

Similarly to (40), we estimate the right side of (46). So for $\delta > 0$, we have:
The first term

$$\begin{aligned}&\int_\Omega \nabla u(t).\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}x \\ &\quad\leqslant \delta\int_\Omega|\nabla u|^2\,\mathrm{d}x + \frac{1}{4\delta}\left[\int_0^t g^{2-p}(\tau)\,\mathrm{d}\tau\right]\int_\Omega\int_0^t g^p(t-\tau)|\nabla u(\tau)-\nabla u(t)|^2\,\mathrm{d}\tau\,\mathrm{d}x. \end{aligned} \tag{47}$$

The second term

$$
\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-s)\nabla u(s)\,\mathrm{d}s\right).\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,\mathrm{d}s\right)\mathrm{d}x \\
&\leqslant \delta \int_\Omega \left|\int_0^t g(t-s)\nabla u(s)\,\mathrm{d}s\right|^2 \mathrm{d}x + \frac{1}{4\delta}\int_\Omega \left|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,\mathrm{d}s\right|^2 \mathrm{d}x \\
&\leqslant \left(2\delta + \frac{1}{4\delta}\right)\int_\Omega \left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|\,\mathrm{d}s\right)^2 \mathrm{d}x + 2\delta(1-l)^2 \int_\Omega |\nabla u|^2\,\mathrm{d}x \\
&\leqslant \left(2\delta + \frac{1}{4\delta}\right)\left[\int_0^t g^{2-p}(\tau)\,\mathrm{d}\tau\right]\int_\Omega\int_0^t g^p(t-\tau)|\nabla u(\tau)-\nabla u(t)|^2\,\mathrm{d}\tau\,\mathrm{d}x \\
&\quad + 2\delta(1-l)^2\int_\Omega |\nabla u|^2\,\mathrm{d}x. \qquad (48)
\end{aligned}
$$

The third term

$$
\begin{aligned}
&\int_\Omega |u|^\gamma u \left(\int_0^t g(t-\tau)(u(t)-u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}x \\
&\leqslant \delta \int_\Omega |u(t)|^{2(\gamma+1)}\,\mathrm{d}x + \frac{1}{4\delta}\int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau))\,\mathrm{d}\tau\right)^2 \mathrm{d}x \\
&\leqslant \delta \int_\Omega |u(t)|^{2(\gamma+1)}\,\mathrm{d}x + \frac{C_p}{4\delta}\left[\int_0^t g^{2-p}(\tau)\,\mathrm{d}\tau\right](g^p\circ\nabla u)(t). \qquad (49)
\end{aligned}
$$

We use (4), (5), (9), and (29) to estimate

$$
\begin{aligned}
\int_\Omega |u(t)|^{2(\gamma+1)}\,\mathrm{d}x &\leqslant C_p^{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} \\
&\leqslant C_p^{2(\gamma+1)}\left(\frac{2(\gamma+2)E(0)}{\gamma l}\right)^\gamma \|\nabla u(t)\|_2^2. \qquad (50)
\end{aligned}
$$

Combining (49) and (50), we get

$$
\begin{aligned}
&\int_\Omega |u|^\gamma u \left(\int_0^t g(t-\tau)(u(t)-u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}x \\
&\leqslant C_p^{2(\gamma+1)}\left(\frac{2(\gamma+2)E(0)}{\gamma l}\right)^\gamma \|\nabla u(t)\|_2^2 + \frac{C_p}{4\delta}\left[\int_0^t g^{2-p}(\tau)\,\mathrm{d}\tau\right](g^p\circ\nabla u)(t). \qquad (51)
\end{aligned}
$$

The fourth term

$$
\begin{aligned}
&-\int_\Omega u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))\,\mathrm{d}\tau\,\mathrm{d}x \\
&\leqslant \delta\int_\Omega |u_t(t)|^2\,\mathrm{d}x + \frac{g(0)}{4\delta}C_p\int_\Omega\int_0^t -g'(t-\tau)|\nabla u(t)-\nabla u(\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau. \qquad (52)
\end{aligned}
$$

A combination of (46)–(52) yields (45). □

Theorem 5.3. *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given and satisfying* (30). *Assume that* (9) *holds and g satisfies* (G1) *and* (G2). *Then, for each* $t_0 > 0$, *there exist positive constants K and k such that the solution of* (1) *satisfies, for all* $t \geqslant t_0$,

$$E(t) \leqslant K\mathrm{e}^{-kt}, \quad p = 1$$
$$E(t) \leqslant K(1+t)^{-1/(p-1)}, \quad p > 1. \tag{53}$$

Proof. Since g is continuous and $g(0) > 0$ then for any $t_0 > 0$ we have

$$\int_0^t g(s)\,\mathrm{d}s \geqslant \int_0^{t_0} g(s)\,\mathrm{d}s = g_0 > 0, \quad \forall t \geqslant t_0. \tag{54}$$

By using (29), (36), (45), and (54), we obtain

$$\begin{aligned} F'(t) \leqslant & -\{\varepsilon_2(g_0 - \delta) - \varepsilon_1\} \int_\Omega u_t^2(t)\,\mathrm{d}x + \varepsilon_1 \|u\|_{\gamma+2}^{\gamma+2} \\ & - \left[\frac{\varepsilon_1 l}{2} - \varepsilon_2 \delta \left\{1 + 2(1-l)^2 + C_p^{2(\gamma+1)} \left(\frac{2(\gamma+2)E(0)}{\gamma l}\right)^\gamma\right\}\right] \|\nabla u\|_2^2 \\ & \times \left(\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_p\right)(g' \circ \nabla u)(t) + \left(\frac{\varepsilon_1}{l} + \varepsilon_2\left\{2\delta + \frac{3}{4\delta}\right\}\right) \\ & \times \left[\int_0^t g^{2-p}(\tau)\,\mathrm{d}\tau\right](g^p \circ \nabla u)(t). \end{aligned} \tag{55}$$

At this point we choose δ so small that

$$g_0 - \delta > \tfrac{1}{2} g_0$$

and

$$\delta \frac{(1 + 2(1-l)^2 + C_p^{2(\gamma+1)}((2(\gamma+2)E(0))/\gamma l)^\gamma)}{l} < \frac{1}{8} g_0.$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\tfrac{1}{4} g_0 \varepsilon_2 < \varepsilon_1 < \tfrac{1}{2} g_0 \varepsilon_2 \tag{56}$$

will make

$$k_1 = \varepsilon_2(g_0 - \delta) - \varepsilon_1 > 0,$$

$$k_2 = \varepsilon_1 \frac{l}{2} - \varepsilon_2 \delta \left\{1 + 2(1-l)^2 + C_p^{2(\gamma+1)} \left(\frac{2(\gamma+2)E(0)}{\gamma l}\right)^\gamma\right\} > 0.$$

We then pick ε_1 and ε_2 so small that (38) and (56) remain valid and

$$k_3 = \frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_p - \frac{1}{\xi}\left(\frac{\varepsilon_1}{l} + \varepsilon_2\left\{2\delta + \frac{3}{4\delta}\right\}\right)\left[\int_0^\infty g^{2-p}(\tau)\,\mathrm{d}\tau\right] > 0.$$

Therefore (55) becomes

$$\begin{aligned} F'(t) \leqslant & -k_1 \int_\Omega u_t^2(t)\,\mathrm{d}x - k_2 \int_\Omega |\nabla u|^2(t)\,\mathrm{d}x \\ & - k_3 \xi (g^p \circ \nabla u)(t) + \varepsilon_1 \|u\|_{\gamma+2}^{\gamma+2}, \quad \forall t \geqslant t_0. \end{aligned} \tag{57}$$

Case 1. $p = 1$:
We combine (38) and (57) to get

$$F'(t) \leqslant -\beta_1 E(t) \leqslant -\beta_1 \alpha_1 F(t) \quad \forall t \geqslant t_0, \tag{58}$$

for some constant $\beta_1 > 0$. A simple integration of (58) leads to

$$F(t) \leqslant F(t_0) \mathrm{e}^{\beta_1 \alpha_1 t_0} \mathrm{e}^{-\beta_1 \alpha_1 t}, \quad \forall t \geqslant t_0. \tag{59}$$

Again by virtue of (38), relation (59) yields

$$E(t) \leqslant \alpha_2 F(t_0) \mathrm{e}^{\beta_1 \alpha_1 t_0} \mathrm{e}^{-\beta_1 \alpha_1 t} = K \mathrm{e}^{-kt}, \quad \forall t \geqslant t_0. \tag{60}$$

Case 2. $p > 1$:
But using (G2) we easily verify that

$$\int_0^\infty g^{1-\theta}(\tau)\,\mathrm{d}\tau < \infty, \quad \forall \theta < 2 - p,$$

so [6, Lemma 3.3] yields

$$(g \circ \nabla u)(t) \leqslant C \left\{ \left(\int_0^\infty g^{1-\theta}(\tau)\,\mathrm{d}\tau \right) E(0) \right\}^{(p-1)/(p-1+\theta)} \{(g^p \circ \nabla u)(t)\}^{\theta/(p-1+\theta)}.$$

Therefore, for any $\sigma > 1$, we get

$$\begin{aligned} E^\sigma(t) &\leqslant C E^{\sigma-1}(0) \left\{ \int_\Omega u_t^2\,\mathrm{d}x - \int_\Omega |u|^{\gamma+2}\,\mathrm{d}x + \|\nabla u\|_2^2 \right\} + C\{(g \circ \nabla u)(t)\}^\sigma \\ &\leqslant C E^{\sigma-1}(0) \left\{ \int_\Omega u_t^2\,\mathrm{d}x - \int_\Omega |u|^{\gamma+2}\,\mathrm{d}x + \|\nabla u\|_2^2 \right\} \\ &\quad + C \left\{ \left(\int_0^\infty g^{1-\theta}(\tau)\,\mathrm{d}\tau \right) E(0) \right\}^{\sigma(p-1)/(p-1+\theta)} \{(g^p \circ \nabla u)(t)\}^{\sigma\theta/(p-1+\theta)}. \end{aligned} \tag{61}$$

By choosing $\theta = \frac{1}{2}$ and $\sigma = 2p - 1$ (hence $\sigma\theta/(p - 1 + \theta) = 1$), estimate (61) gives

$$E^\sigma(t) \leqslant C \left\{ \int_\Omega u_t^2\,\mathrm{d}x - \int_\Omega |u|^{\gamma+2}\,\mathrm{d}x + \|\nabla u\|_2^2 + (g^p \circ \nabla u)(t) \right\}. \tag{62}$$

By combining (38), (57) and (62), we obtain

$$F'(t) \leqslant -\frac{\beta_2}{\Gamma} E^\sigma(t) \leqslant -\frac{\beta_2}{\Gamma} (\alpha_1)^\sigma F^\sigma(t), \quad \forall t \geqslant t_0, \tag{63}$$

for some constant $\beta_2 > 0$. A simple integration of (63) over (t_0, t) leads to

$$F(t) \leqslant C_1 (1 + t)^{-1/(\sigma-1)}, \quad \forall t \geqslant t_0. \tag{64}$$

As a consequence of (64), we easily verify that

$$\int_0^\infty F(t)\,\mathrm{d}t + \sup_{t \geqslant 0} t F(t) < \infty.$$

Therefore, by using Lemma 3.3 of [6], we have

$$g \circ \nabla u \leqslant C_2 \left[\int_0^t \|u(s)\|_{H^1(0,1)} \, ds + t\|u(t)\|_{H^1(0,1)} \right]^{(p-1)/p} (g^p \circ \nabla u)^{1/p}$$
$$\leqslant C_2 \left[\int_0^t F(s) \, ds + tF(t) \right]^{(p-1)/p} (g^p \circ \nabla u)^{1/p} \leqslant C_3 (g^p \circ \nabla u)^{1/p},$$

which implies that

$$g^p \circ \nabla v \geqslant C_4 (g \circ \nabla u)^p. \tag{65}$$

Consequently, a combination of (57) and (65) yields

$$F'(t) \leqslant -C_5 \left[\int_\Omega u_t^2(t) \, dx + \int_\Omega |\nabla u|^2(t) \, dx + (g \circ \nabla u)^p(t) - \|u\|_{\gamma+2}^{\gamma+2} \right], \quad \forall t \geqslant t_0. \tag{66}$$

On the other hand, we have, similarly to (61),

$$E^p(t) \leqslant C_6 \left[\int_\Omega u_t^2(t) \, dx + \int_\Omega |\nabla u|^2(t) \, dx + (g \circ \nabla u)^p(t) - \|u\|_{\gamma+2}^{\gamma+2} \right], \quad \forall t \geqslant t_0. \tag{67}$$

Combining the last two inequalities and (38), we obtain

$$F'(t) \leqslant -C_7 F^p(t), \quad t \geqslant t_0.$$

A simple integration of (68) over (t_0, t) gives

$$F(t) \leqslant K(1+t)^{-1/(p-1)}, \quad t \geqslant t_0.$$

This completes the proof. □

Remark 5.1. Note that our result is proved without any condition on g'' and g'''. Unlike what was assumed in (2.4) of [12], we only need g to be differentiable and satisfying (G1) and (G2).

Remark 5.2. Estimates (53) hold for $t \geqslant 0$ by virtue of (60), (69), and the boundedness of $E(t)$ on $[0, t_0]$.

Acknowledgements

The authors would like to express their sincere thanks to King Fahd University of Petroleum and Minerals for its support. This work has been funded by KFUPM under Project # MS/ VISCO ELASTIC 270.

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