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Blow Up in a Nonlinearly Damped Wave Equation

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Abstract. In this paper we consider the nonlinearly damped semilinear wave equation

$$u_{tt} - \Delta u + a u_t \, |u_t|^{m-2} = b u |u|^{p-2}$$

associated with initial and Dirichlet boundary conditions. We prove that any strong solution, with negative initial energy, blows up in finite time if $p > m$. This result improves an earlier one in [2].

1. Introduction

In this paper we are concerned with the following initial boundary value problem.

$$\begin{aligned} & u_{tt} - \Delta u + a u_t \, |u_t|^{m-2} = b u \, |u|^{p-2} , \quad x \in \Omega , \quad t > 0 , \\ & u(x,t) = 0 , \quad x \in \partial\Omega , \quad t \geq 0 , \\ & u(x,0) = u_0(x) , \quad u_t(x,0) = u_1(x) , \quad x \in \Omega , \end{aligned} \tag{1.1}$$

where $a, b > 0$, $p, m > 2$, and Ω is a a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. For $b = 0$, it is well–known that the damping term $a u_t \, |u_t|^{m-2}$ assures global existence for arbitrary initial data (see [3], [5]). If $a = 0$ then the source term $b u \, |u|^{p-2}$ causes finite time blow up of solutions with negative initial energy (see [1], [4], [6], [7]).

The interaction between the damping and the source terms was first considered by Levine [6], [7] in the linear damping case ($m = 2$). He showed that solutions with negative initial energy blow up in finite time. Recently Georgiev and Todorova [2] extended Levine's result to the nonlinear case ($m > 2$). In their work, the authors introduced a different method and determined suitable relations between m and p, for which there is global existence or alternatively finite time blow up. Precisely: they showed that solutions with negative energy continue to exist globally "in time"

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if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative.

This result has been lately generalized to an abstract setting and to unbounded domains by Levine and Serrin [8] and Levine, Park, and Serrin [9]. In these papers, the authors showed that no solution with negative energy can be extended on $[0, \infty)$ if $p > m$ and proved several noncontinuation theorems. This generalization allowed them also to apply their result to quasilinear situations, of which problem (1.1) is a particular case.

Vitillaro [10] combined the arguments in [2] and [8] to extend these results to situations where the damping is nonlinear and the solution has positive initial energy.

In this work, we prove the same result of [2] without imposing the condition that the initial energy is sufficiently negative. In other words, we show that any solution of (1.1) with negative initial energy — however close to zero is — blows up in finite time. In addition to ommitting the condition of large "negative" initial data, our technique of proof is simpler than the ones in [2] and [8]. We first state a local result established in [2].

Theorem 1.1. *Suppose that $m > 2$, $p > 2$, and*

$$p \leq 2\frac{n-1}{n-2}, \quad n \geq 3. \tag{1.2}$$

Assume further that

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega). \tag{1.3}$$

Then the problem (1.1) *has a unique local solution*

$$u \in C\big([0,T); H_0^1(\Omega)\big), \quad u_t \in C\big([0,T); L^2(\Omega)\big) \cap L^m(\Omega \times (0,T)), \tag{1.4}$$

T is small.

Remark 1.2. The condition on p, in (1.2), is needed to establish the local existence result (see [2]). In fact under this condition, the nonlinearity is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$.

2. Main result

In this section we show that the solution (1.4) blows up in finite time if $p > m$ and $E(0) < 0$, where

$$E(t) := \frac{1}{2}\int_\Omega \left[u_t^2 + |\nabla u|^2\right](x,t)\,dx - \frac{b}{p}\int_\Omega |u(x,t)|^p\,dx. \tag{2.1}$$

Lemma 2.1. *Suppose that* (1.2) *holds. Then there exists a positive constant $C > 1$ depending on Ω only such that*

$$||u||_p^s \leq C\big(||\nabla u||_2^2 + ||u||_p^p\big) \tag{2.2}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

Proof. If $||u||_p \leq 1$ then $||u||_p^s \leq ||u||_p^2 \leq C\,||\nabla u||_2^2$ by Sobolev embedding theorems. If $||u||_p > 1$ then $||u||_p^s \leq ||u||_p^p$. Therefore (2.2) follows. □

We set

$$H(t) \;:=\; -E(t)$$

and use, throughout this paper, C to denote a generic positive constant depending on Ω only. As a result of (2.1), (2.2), we have

Corollary 2.2. *Let the assumptions of the lemma hold. Then we have*

$$||u||_p^s \;\leq\; C\big(|H(t)| + ||u_t||_2^2 + ||u||_p^p\big) \tag{2.3}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p$.

Theorem 2.3. *Let the conditions of the Theorem* 1.1 *be fulfilled. Assume further that* $p > m$ *and*

$$E(0) \;<\; 0\,. \tag{2.4}$$

Then the solution (1.4) *blows up in finite time.*

Remark 2.4. Note that contrary to [2], no condition on the size of the initial data has been done. The blow up takes place for any initial data satisfying (2.4).

Proof. We multiply Equation (1.1) by u_t and integrate over Ω to get

$$E'(t) \;=\; -a\int_\Omega |u_t(x,t)|^m\,dx\,, \tag{2.5}$$

for almost every t in $[0,T)$ since $E'(t)$ is absolutely continuous (see [2]); hence $H'(t) \geq 0$. So we have

$$0 \;<\; H(0) \;\leq\; H(t) \;\leq\; \frac{b}{p}\,||u||_p^p\,, \tag{2.6}$$

for every t in $[0,T)$, by virtue of (2.4). We then define

$$L(t) \;:=\; H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t(x,t)\,dx \tag{2.7}$$

for ε small to be chosen later and

$$0 \;<\; \alpha \;\leq\; \min\left\{\frac{(p-2)}{2p}, \frac{(p-m)}{p(m-1)}\right\}\,. \tag{2.8}$$

By taking a derivative of (2.7) and using Equation (1.1) we obtain

$$\begin{aligned} L'(t) \;:=\; & (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega \big[u_t^2 - |\nabla u|^2\big](x,t)\,dx \\ & + \varepsilon b\int_\Omega |u(x,t)|^p\,dx - a\varepsilon\int_\Omega |u_t|^{m-2}u_t u(x,t)\,dx\,. \end{aligned} \tag{2.9}$$

We then exploit Young's inequality

$$XY \;\le\; \frac{\delta^r}{r}\,X^r + \frac{\delta^{-q}}{q}\,Y^q, \quad X\,,\,Y \;\ge\; 0\,, \quad \text{for all} \quad \delta \;>\; 0\,, \quad \frac{1}{r}+\frac{1}{q} \;=\; 1$$

with $r = m$ and $q = m/(m-1)$ to estimate the last term in (2.9) as follows

$$\int_\Omega |u_t|^{m-1}\,|u|\,dx \;\le\; \frac{\delta^m}{m}\,||u||_m^m + \frac{m-1}{m}\,\delta^{-m/(m-1)}\,||u_t||_m^m$$

which yields, by substitution in (2.9),

$$\begin{aligned} L'(t) \;\ge\;& \left[(1-\alpha)H^{-\alpha}(t) - \frac{m-1}{m}\,\varepsilon\delta^{-m/(m-1)}\right]H'(t) \\ &+ \varepsilon\int_\Omega \left[u_t^2 - |\nabla u|^2\right](x,t)\,dx + \varepsilon\left[pH(t) + \frac{p}{2}\int_\Omega \left[u_t^2 + |\nabla u|^2\right](x,t)\,dx\right] \\ &- \varepsilon a\,\frac{\delta^m}{m}\,||u||_m^m, \quad \text{for all} \quad \delta \;>\; 0\,. \end{aligned} \tag{2.10}$$

Of course (2.10) remains valid even if δ is time dependant since the integral is taken over the x variable. Therefore by taking δ so that $\delta^{-m/(m-1)} = kH^{-\alpha}(t)$, for large k to be specified later, and substituting in (2.10) we arrive at

$$\begin{aligned} L'(t) \;\ge\;& \left[(1-\alpha) - \frac{m-1}{m}\,\varepsilon k\right]H^{-\alpha}(t)H'(t) + \varepsilon\Big(\frac{p}{2}+1\Big)\int_\Omega u_t^2(x,t)\,dx \\ &+ \varepsilon\Big(\frac{p}{2}-1\Big)\int_\Omega |\nabla u|^2(x,t)\,dx + \varepsilon\left[pH(t) - \frac{k^{1-m}}{m}\,aH^{\alpha(m-1)}(t)\,||u||_m^m\right]. \end{aligned} \tag{2.11}$$

By exploiting (2.6) and the inequality $||u||_m^m \le C\,||u||_p^m$, we obtain

$$H^{\alpha(m-1)}(t)\,||u||_m^m \;\le\; \left(\frac{b}{p}\right)^{\alpha(m-1)} C\,||u||_p^{m+\alpha p(m-1)}\,,$$

hence (2.11) yields

$$\begin{aligned} L'(t) \;\ge\;& \left[(1-\alpha) - \frac{m-1}{m}\,\varepsilon k\right]H^{-\alpha}(t)H'(t) + \varepsilon\Big(\frac{p}{2}+1\Big)\int_\Omega u_t^2(x,t)\,dx \\ &+ \varepsilon\Big(\frac{p}{2}-1\Big)\int_\Omega |\nabla u|^2(x,t)\,dx \\ &+ \varepsilon\left[pH(t) - \frac{k^{1-m}}{m}\,a\left(\frac{b}{p}\right)^{\alpha(m-1)} C\,||u||_p^{m+\alpha p(m-1)}\right]. \end{aligned} \tag{2.12}$$

We then use Corollary 2.2 and (2.8), for $s = m+\alpha p(m-1) \le p$, to deduce from (2.12)

$$\begin{aligned} L'(t) \;\ge\;& \left[(1-\alpha) - \frac{m-1}{m}\,\varepsilon k\right]H^{-\alpha}(t)H'(t) + \varepsilon\Big(\frac{p}{2}+1\Big)\int_\Omega u_t^2(x,t)\,dx \\ &+ \varepsilon\Big(\frac{p}{2}-1\Big)\int_\Omega |\nabla u|^2(x,t)\,dx \\ &+ \varepsilon\big[pH(t) - C_1k^{1-m}\big\{H(t) + ||u_t||_2^2 + ||u||_p^p\big\}\big]\,, \end{aligned} \tag{2.13}$$

where $C_1 = a\left(\frac{b}{p}\right)^{\alpha(m-1)} C/m$. By noting that

$$H(t) \;=\; \frac{b}{p}\,||u||_p^p - \frac{1}{2}\,||u_t||_2^2 - \frac{1}{2}\,||\nabla u||_2^2$$

and writing $p = (p+2)/2 + (p-2)/2$, (2.13) yields

$$\begin{aligned} L'(t) \;\geq\; & \left[(1-\alpha) - \frac{m-1}{m}\,\varepsilon k\right] H^{-\alpha}(t)H'(t) + \frac{p-2}{4}\,||\nabla u||_2^2 \\ & + \varepsilon\left[\left(\frac{p+2}{2} - C_1 k^{1-m}\right) H(t) + \left(\frac{p-2}{2p} b - C_1 k^{1-m}\right) ||u||_p^p \right. \\ & \left. + \left(\frac{p+6}{4} - C_1 k^{1-m}\right) ||u_t||_2^2 \right] \end{aligned} \tag{2.14}$$

At this point, we choose k large enough so that the coefficients of $H(t)$, $||u_t||_2^2$, and $||u||_p^p$ in (2.14) are strictly positive; hence we get

$$L'(t) \;\geq\; \left[(1-\alpha) - \frac{m-1}{m}\,\varepsilon k\right] H^{-\alpha}(t)H'(t) + \varepsilon\gamma\big[H(t) + ||u_t||_2^2 + ||u||_p^p\big]\,, \tag{2.15}$$

where $\gamma > 0$ is the minimum of these coefficients. Once k is fixed (hence γ), we pick ε small enough so that $(1-\alpha) - \varepsilon k(m-1)/m \geq 0$ and

$$L(0) \;=\; H^{1-\alpha}(0) + \varepsilon \int_\Omega u_0 u_1(x)\,dx \;>\; 0\,.$$

Therefore (2.15) takes the form

$$L'(t) \;\geq\; \gamma\varepsilon\left[H(t) + ||u_t||_2^2 + ||u||_p^p\right]. \tag{2.16}$$

Consequently we have

$$L(t) \;\geq\; L(0) \;>\; 0\,, \quad \text{for all} \quad t \;\geq\; 0\,.$$

Next we would like to show that

$$L'(t) \;\geq\; \Gamma L^{1/(1-\alpha)}(t)\,, \quad \text{for all} \quad t \;\geq\; 0\,, \tag{2.17}$$

where Γ is a positive constant depending on $\varepsilon\gamma$ and C (the constant of Lemma 2.1). Once (2.17) is established, we obtain in a standard way the finite time blow up of $L(t)$, hence of u (see [1] for instance).

To prove (2.17), we first estime

$$\left|\int_\Omega u u_t(x,t)\,dx\right| \;\leq\; ||u||_2\,||u_t||_2 \;\leq\; C\,||u||_p\,||u_t||_2$$

which implies

$$\left|\int_\Omega u u_t(x,t)\,dx\right|^{1/(1-\alpha)} \;\leq\; C\,||u||_p^{1/(1-\alpha)}\,||u_t||_2^{1/(1-\alpha)}\,.$$

Again Young's inequality gives us

$$\left|\int_\Omega u u_t(x,t)\,dx\right|^{1/(1-\alpha)} \le C\left[||u||_p^{\mu/(1-\alpha)} + ||u_t||_2^{\theta/(1-\alpha)}\right], \tag{2.18}$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\alpha)$, to get $\mu/(1-\alpha) = 2/(1-2\alpha) \le p$ by (2.8). Therefore (2.18) becomes

$$\left|\int_\Omega u u_t(x,t)\,dx\right|^{1/(1-\alpha)} \le C\left[||u||_p^s + ||u_t||_2^2\right],$$

where $s = 2/(1-2\alpha) \le p$. By using Corollary 2.2 we obtain

$$\left|\int_\Omega u u_t(x,t)\,dx\right|^{1/(1-\alpha)} \le C\left[H(t) + ||u||_p^p + ||u_t||_2^2\right], \quad \text{for all} \quad t \ge 0. \tag{2.19}$$

Finally by noting that

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t(x,t)\,dx\right)^{1/(1-\alpha)} \\ &\le 2^{1/(1-\alpha)}\left(H(t) + \left|\int_\Omega u u_t(x,t)\,dx\right|^{1/(1-\alpha)}\right) \end{aligned}$$

and combining it with (2.16) and (2.19), the inequality (2.17) is established. This completes the proof. □

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