Midterm 1 / Math 531

Academic Year 2005/06/052

problem # 1

We discuss two cases

$a = 0$

(i) $\sup_{k \geq n} x_k = \frac{1}{n^2} \Rightarrow \limsup_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1}{n^2} = 0$

$\inf_{k \geq n} x_k = 0 \Rightarrow \liminf_{n \to \infty} x_n = 0$

(ii) In this case (x_n) converges such that $\lim_{n \to \infty} x_n = 0$

$a > 0$

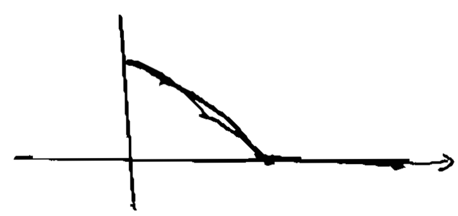
(i) $\sup_{k \geq n} x_k = a, \quad \forall n > \frac{1}{\sqrt{a}}$.

So $\limsup_{n \to \infty} x_n = a > 0$

$\inf_{k \geq n} x_k = 0 \Rightarrow \liminf_{n \to \infty} x_n = 0$

(ii) In this case (x_n) diverges.

problem # 2



(i) $\lim_{x \xrightarrow{<} 1} f_n(x) = \lim_{x \to 1} (1 - x^n) = 0$

$\lim_{x \xrightarrow{>} 1} f_n(x) = \lim_{x \to 1} 0 = 0$

So f_n is continuous at $x = 1$. In fact, f_n continuous on $[0, 2]$.

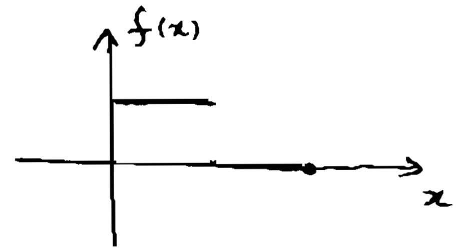
(ii) $f_n(x) = 0, \quad 1 \leq x \leq 2 \Rightarrow f(x) = \lim_{n \to \infty} f_n(x) = 0, \quad 1 \leq x \leq 2.$

$f_n(x) = 1 - x^n, \quad 0 \leq x < 1 \Rightarrow f(x) = \lim_{n \to \infty} (1 - x^n) = 1, \quad 0 \leq x < 1$

(iii)

f is discontinuous at $x=1$

therefore the Convergence is not uniform.



Problem # 3 It is clear that $G_{n+1} \subset G_n$

$$G^k = \bigcup_{n=1}^{k} G_n = G_1 = (0, \tfrac{1}{2}) \Rightarrow m G^k = \tfrac{1}{2}$$

$$\Rightarrow G = \lim_{k \to \infty} G^k \Rightarrow m G = \tfrac{1}{2}.$$

Also $m G_1 = \frac{1}{2} < +\infty$

So $m F = \lim\limits_{n \to \infty} G_n = \lim\limits_{n \to \infty} \frac{1}{2^n} = 0.$

(ii) Since $mE < +\infty$ and since $m^* E = mE$ then there exists a covering of open intervals (I_n) such that $E = \bigcup_{n=1}^{\infty} I_n$ and $m^* E + \varepsilon > \sum_{n=1}^{\infty} \ell(I_n)$

Take $\mathcal{O} = \bigcup_{n=1}^{\infty} I_n$. So, $\mathcal{O}$ is open and

$$m\mathcal{O} = m^* \mathcal{O} \leq \sum_{n=1}^{\infty} \ell(I_n) < mE + \varepsilon$$

$$\Rightarrow m\mathcal{O} - mE < \varepsilon.$$

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problem # 4

(i) If $x \in A$ then $f(x) < g(x) \Rightarrow \exists\, \alpha_0 \in \mathbb{Q} \,/\, f(x) < \alpha_0 < g(x) \Rightarrow$ $\{x \in D \,/\, f(x) < \alpha_0\} \cap \{x \in D \,/\, g(x) > \alpha_0\} = A_{\alpha_0} \Rightarrow x \in \bigcup_{\alpha \in \mathbb{Q}} A_\alpha$

Now, if $x \in \bigcup_{\alpha \in \mathbb{Q}} A_\alpha \Rightarrow \exists\, \alpha \in \mathbb{Q} \,/\, f(x) < \alpha < g(x) \Rightarrow x \in A$

hence $A = \bigcup_{\alpha \in \mathbb{Q}} A_\alpha$.

(ii) Since f and g are measurable then $\{x \in D \,/\, f(x) < \alpha\}$ and $\{x \in D \,/\, \alpha < g(x)\}$ are measurable $\Rightarrow A_\alpha$ is measurable

So, $\bigcup_{\alpha \in \mathbb{Q}} A_\alpha$ is measurable.

problem # 5

(i) Define $E_n = \{x \in E \,/\, |f(x)| > n\}$, $n = 0, 1, 2, \ldots$

It is clear that $E_{n+1} \subset E_n$ since, if $x \in E_{n+1} \Rightarrow$

$|f(x)| > n+1 > n \Rightarrow x \in E_n$.

Also $mE_n \leq mE_0 < +\infty$, $\forall n$.

So $m\left(\bigcap_{n=0}^{\infty} E_n\right) = \lim_{n\to\infty} mE_n$

But $\bigcap_{n=0}^{\infty} E_n = \phi$, since $\forall x \in \bigcap_{n=0}^{\infty} E_n \Rightarrow |f(x)| > n, \forall n$

$\Rightarrow f(x) \notin \mathbb{R}$, which not true. Thus $\bigcap_{n=0}^{\infty} E_n = \phi$

hence $\lim_{n\to\infty} mE_n = 0 \Rightarrow$

$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \,/\, \forall n \geq N, \; mE_n < \varepsilon$

Take $A = E_N \Rightarrow mA < \varepsilon$ and $|f(x)| \leq N$, $\forall x \in E \setminus A$

(ii) let $E = (0, +\infty)$ and $f(x) = x$

In this case $\forall M > 0$, $\{x \in E \,/\, |f(x)| > M\} = (M, +\infty)$

$\Rightarrow mA = +\infty > \varepsilon$, $\forall \varepsilon > 0$.