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Blow up and global existence in a nonlinear viscoelastic wave equation

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In this paper the nonlinear viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\, d\tau + a u_t\, |u_t|^{m-2} \;=\; bu\,|u|^{p-2}$$

associated with initial and Dirichlet boundary conditions is considered. Under suitable conditions on g, it is proved that any weak solution with negative initial energy blows up in finite time if $p > m$. Also the case of a stronger damping is considered and it is showed that solutions exist globally for any initial data, in the appropriate space, provided that $m \geq p$.

1 Introduction

In this paper we are concerned with the following initial boundary value problem .

$$\begin{aligned} & u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\, d\tau + a u_t\, |u_t|^{m-2} \;=\; bu\,|u|^{p-2}\,, \quad x \in \Omega\,, \quad t \;>\; 0\,, \\ & u(x,t) \;=\; 0\,, \quad x \in \partial\Omega\,, \quad t \;\geq\; 0\,, \\ & u(x,0) \;=\; u_0(x)\,, \quad u_t(x,0) \;=\; u_1(x)\,, \quad x \in \Omega\,, \end{aligned} \tag{1.1}$$

where $a, b > 0$, $p > 2$, $m \geq 1$, and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. In the absence of the viscoelastic term $(g = 0)$, the problem has been extensively studied and results concerning existence and nonexistence have been established. For $a = 0$, the source term $bu\,|u|^{p-2}$ causes finite time blow up of solutions with negative initial energy (see [2], [8]). For $b = 0$, the damping term $au_t\,|u_t|^{m-2}$ assures global existence for arbitrary initial data (see [7], [9]). The interaction between the damping and the source terms was first considered by Levine [10], [11] in the linear damping case $(m = 2)$. He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [6] extended Levine's result to the nonlinear damping case $(m > 2)$. In their work, the authors introduced a different method and determined suitable relations between m and p for which there is global existence or alternatively finite time blow up. More precisely: they showed that solutions with any initial data continue to exist globally "in time" if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative. Without imposing the condition that the initial energy is sufficiently negative, Messaoudi [17] extended the blow up result of [6] to solutions with negative initial energy only. For results of the same nature, we refer the reader to Levine and Serrin [12], Levine, Park, and Serrin [13], and Vitillaro [19].

In the presence of the viscoelastic term $(g \neq 0)$, Cavalcanti *et al.* [4] studied (1.1) for $m = 2$, and a localized damping $a(x)u_t$ ($a(x)$ can be null on a part of the boundary). They obtained an exponential rate of decay by assuming that the kernel g is of exponential decay. This work extended the result of Zuazua [20] in which he considered (1.1) with $g = 0$ and the linear damping is localized. When the damping is caused only by the memory

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term ($a = 0$), an exponential decay result can be obtained, at least for small initial data, by following the idea of proof established by Monuz Rivera in [18]. In this paper, Monuz Rivera proved that the first and the second-order energies of the solution to a viscoelastic plate, decay exponentially provided that the kernel of the memory decays exponentially. In the same direction, Cavalcanti *et al* [3] have also studied the following system

$$|u_t|^\rho\, u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\, d\tau - \gamma \Delta u_t \;=\; 0\,, \quad x \in \Omega\,,\;\; t \;>\; 0\,,$$

$\rho > 0$. They proved a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. Related to our work, we also mention the work of Wei J. Liu [15] in which he used the multiplier techniques to establish an exponential decay result in the higher dimensional thermoeviscoelasticity. The same method was also used in [16] to prove, under appropriate conditions on the coupling parameters and relaxation function, a partial exact controllability result for a linear thermoviscoelastic model. These last results generalize earlier ones [14] established for thermoelasticity.

Finally, it is also worth mentionning the work of Aassila *et al* [1] and Cavalcanti *et al* [5]. In his work, Aaassila established an asymptotic stability and decay rates, for solutions of the wave equation in star-shaped domains, were established by combination of memory effect and damping mechanism. In [5], an existence and decay result for viscoelastic problems with nonlinear boundary damping has been proved.

In this article, we establish a blow up result for solutions with negative initial energy and $m < p$. Our technique of proof is similar to the one in [17] with some necessary modifications due the nature of the problem treated here. We also prove global existence for arbitrary initial data (in the appropriate space) if $m \geq p$.

We first state a local result existence theorem which can be established by combination of the arguments in [3] and [6].

Theorem 1.1 *Suppose that $m \geq 1$, $p > 2$ and let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume further that*

$$\max\{m, p\} \;\leq\; \frac{2(n-1)}{n-2}\,, \quad n \;\geq\; 3 \tag{1.2}$$

and g is a C^1 function satisfying

$$1 - \int_0^\infty g(s)\, ds \;=\; l \;>\; 0\,. \tag{1.3}$$

Then problem (1.1) *has a unique local solution*

$$u \in C\big([0, T_m); H_0^1(\Omega)\big)\,, \quad u_t \in C\big([0, T_m); L^2(\Omega)\big) \cap L^{m+1}(\Omega \times 0, T_m)\big)\,, \tag{1.4}$$

for some $T_m > 0$.

Remark 1.2 Condition (1.2) is needed to establish the local existence result (see [3], [6]). In fact under this condition, the nonlinearity is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$. Condition (1.3) is necessary to guarantee the hyperbolicity and well-posedness of the system (1.1).

2 Blow up

In this section we state and prove our main result. For this purpose we define

$$E(t) \;=\; \frac{1}{2}\,||u_t||_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right)||\nabla u||_2^2 + \frac{1}{2}\,(g \circ \nabla u)(t) - \frac{b}{p}\,||u||_p^p\,, \tag{2.1}$$

where

$$(g \circ v)(t) \;=\; \int_0^t g(t-\tau)\, ||v(t) - v(\tau)||_2^2\, d\tau$$

and make the following extra assumptions on g

$$g(s) \;\geq\; 0\,, \quad g'(s) \;\leq\; 0\,, \quad \int_0^\infty g(s)\, ds \;<\; \frac{(p/2)-1}{(p/2)-1+(1/2p)}\,. \tag{2.2}$$

Theorem 2.1 *Suppose that* $m > 1, p > \max\{2, m\}$ *satisfying* (1.2). *Assume further that* (2.2) *holds and*

$$E_0 \;=\; \frac{1}{2}\,||u_1||_2^2 + \frac{1}{2}\,||\nabla u_0||_2^2 - \frac{b}{p}\,||u_0||_p^p \;<\; 0\,. \tag{2.3}$$

Then the solution (1.4) *blows up in finite time*

$$T^* \;\leq\; \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}}\,, \tag{2.4}$$

where Γ *and* α *are positive constant with* $\alpha < 1$ *and* L *is given by* (2.12) *below.*

Remark 2.2 By following the steps of the proof of Theorem 2.1 closely, one can easily see that the blow-up result holds even for $m = 1$ (damping caused only by viscosity). A small modification is needed in the proof.

Remark 2.3 A careful examination of the proof shows that a similar result can be established without condition $(2.2)_3$ provided that E_0 is sufficiently negative and $\int_0^\infty g(s)\,ds < 1$.

Remark 2.4 Condition $(2.2)_3$ shows that there is a strong relation between the nonlinearity in the source and the damping caused by the viscosity. More precisely the closer the value of $\int_0^\infty g(s)\,ds$ to 1, the larger p should be in order to guarantee the blow up.

In order to carry the proof of Theorem 2.1, we need the following

Lemma 2.5 *Suppose that* (1.2) *holds. Then there exists a positive constant* $C > 1$ *depending on* Ω *only such that*

$$||u||_p^s \;\leq\; C\left(||\nabla u||_2^2 + ||u||_p^p\right) \tag{2.5}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p$.

P r o o f. If $||u||_p \leq 1$ then $||u||_p^s \leq ||u||_p^2 \leq C\,||\nabla u||_2^2$ by Sobolev embedding theorems. If $||u||_p > 1$ then $||u||_p^s \leq ||u||_p^p$. Therefore (2.5) follows. □

We set

$$H(t) \;:=\; -E(t)$$

and use, throughout this paper, C to denote a generic positive constant depending on Ω only. As a result of (2.1) and (2.5), we have

Corollary 2.6 *Let the assumptions of the lemma hold. Then we have the following*

$$||u||_p^s \;\leq\; C\big(-H(t) - ||u_t||_2^2 - (g \circ \nabla u)(t) + ||u||_p^p\big)\,, \quad \textit{for all} \quad t \in [0, T)\,, \tag{2.6}$$

for any $u(.,t) \in H_0^1(\Omega)$ *and* $2 \leq s \leq p$.

P r o o f of Theorem 2.1. By multiplying equation (1.1) by $-u_t$ and integrating over Ω we obtain

$$\begin{aligned} &\frac{d}{dt}\left\{-\frac{1}{2}\int_\Omega |\nabla u_t|^2\,dx - \frac{1}{2}\int_\Omega |u_t|^2\,dx + \frac{b}{p}\int_\Omega |u|^p\,dx\right\} \\ &\qquad + \int_0^t g(t-\tau)\int_\Omega \nabla u_t(t).\nabla u(\tau)\,dx\,d\tau \;=\; a\int_\Omega |u_t|^m\,dx\,, \end{aligned} \tag{2.7}$$

for any regular solution. This result can be extended to weak solutions by density argument. But

$$\begin{aligned} &\int_0^t g(t-\tau)\int_\Omega \nabla u_t(t).\nabla u(\tau)\,dx\,d\tau \\ &= \int_0^t g(t-\tau)\int_\Omega \nabla u_t(t).[\nabla u(\tau) - \nabla u(t)]\,dx\,d\tau + \int_0^t g(t-\tau)\int_\Omega \nabla u_t(t).\nabla u(t)\,dx\,d\tau \;= \end{aligned} \tag{2.8}$$

$$= -\frac{1}{2}\int_0^t g(t-\tau)\,\frac{d}{dt}\int_\Omega |\nabla u(\tau)-\nabla u(t)|^2\,dx\,d\tau + \int_0^t g(\tau)\left(\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla u(t)|^2\,dx\right)d\tau$$
$$= -\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-\tau)\int_\Omega |\nabla u(\tau)-\nabla u(t)|^2\,dx\,d\tau\right] + \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(\tau)\int_\Omega |\nabla u(t)|^2\,dx\,d\tau\right]$$
$$+ \frac{1}{2}\int_0^t g'(t-\tau)\int_\Omega |\nabla u(\tau)-\nabla u(t)|^2\,dx\,d\tau - \frac{1}{2}g(t)\int_\Omega |\nabla u(t)|^2\,dx\,d\tau\,.$$

We then insert (2.8) in (2.7) to get

$$\begin{aligned}
&\frac{d}{dt}\left\{-\frac{1}{2}\int_\Omega |\nabla u_t|^2\,dx - \frac{1}{2}\int_\Omega |u_t|^2\,dx + \frac{b}{p}\int_\Omega |u|^p\,dx\right\}\\
&-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-\tau)\int_\Omega |\nabla u(\tau)-\nabla u(t)|^2\,dx\,d\tau\right] + \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(\tau)\,||\nabla u(t)||^2\,d\tau\right]\\
&= a\int_\Omega |u_t|^m\,dx - \frac{1}{2}\int_0^t g'(t-\tau)\int_\Omega |\nabla u(\tau)-\nabla u(t)|^2\,dx\,d\tau + \frac{1}{2}g(t)\,||\nabla u(t)||^2\,.
\end{aligned} \tag{2.9}$$

By using the definition of $H(t)$, the estimate (2.9) becomes

$$H'(t) = a\int_\Omega |u_t|^m\,dx - \frac{1}{2}(g'\circ\nabla u)(t) + \frac{1}{2}g(t)\,||\nabla u(t)||^2 \geq 0\,. \tag{2.10}$$

Consequently we have

$$0 < H(0) \leq H(t) \leq \frac{b}{p}\,||u||_p^p\,, \tag{2.11}$$

by virtue of (2.1), (2.10). We then define

$$L(t) := H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t(x,t)\,dx \tag{2.12}$$

for ε small to be chosen later and

$$0 < \alpha \leq \min\left\{\frac{(p-2)}{2p}, \frac{(p-m)}{p(m-1)}\right\}\,. \tag{2.13}$$

By taking a derivative of (2.12) and using Equation (1.1) we obtain

$$\begin{aligned}
L'(t) &= (1-\alpha)H^{-\alpha}(t)\left\{a\,||u_t||_m^m - \frac{1}{2}(g'\circ\nabla u)(t) + \frac{1}{2}g(t)\,||\nabla u||_2^2\right\}\\
&\quad + \varepsilon\int_\Omega \left[u_t^2 - |\nabla u|^2\right](x,t)\,dx + \varepsilon\int_0^t g(t-\tau)\int_\Omega \nabla u(t).\nabla u(\tau)\,dx\,d\tau\\
&\quad + \varepsilon b\int_\Omega |u(x,t)|^p\,dx - a\varepsilon\int_\Omega |u_t|^{m-2}\,u_t u(x,t)\,dx\\
&\geq a(1-\alpha)H^{-\alpha}(t)\,||u_t||_m^m + \varepsilon\int_\Omega \left[u_t^2 - |\nabla u|^2\right](x,t)\,dx\\
&\quad + \varepsilon b\int_\Omega |u(x,t)|^p\,dx - a\varepsilon\int_\Omega |u_t|^{m-2}\,u_t u(x,t)\,dx\\
&\quad + \varepsilon\int_0^t g(t-\tau)\int_\Omega \nabla u(t).[\nabla u(\tau)-\nabla u(t)]\,dx\,d\tau + \varepsilon\int_0^t g(t-\tau)\,||\nabla u(t)||_2^2\,d\tau\,.
\end{aligned} \tag{2.14}$$

By using Schwarz inequality, (2.14) takes the form

$$\begin{aligned} L'(t) \;\ge\; & a(1-\alpha)H^{-\alpha}(t)\,||u_t||_m^m + \varepsilon \int_\Omega \left[u_t^2 - |\nabla u|^2\right](x,t)\,dx \\ & +\; \varepsilon b \int_\Omega |u(x,t)|^p\,dx - a\varepsilon \int_\Omega |u_t|^{m-2} u_t u(x,t)\,dx \\ & -\; \varepsilon \int_0^t g(t-\tau)\,||\nabla u(t)||_2\,||\nabla u(\tau) - \nabla u(t)||_2\,d\tau \\ & +\; \varepsilon \int_0^t g(t-\tau)\,||\nabla u(t)||_2^2\,d\tau\,. \end{aligned} \tag{2.15}$$

We then exploit Young's inequality to estimate the fifth term in the RHS of (2.15) and use (2.1) to substitute for $b\int_\Omega |u(x,t)|^p\,dx$; hence (2.15) becomes

$$\begin{aligned} L'(t) \;\ge\; & a(1-\alpha)H^{-\alpha}(t)\,||u_t||_m^m + \varepsilon \int_\Omega u_t^2(x,t)\,dx \\ & -\; \varepsilon\Big(1 - \int_0^t g(s)\,ds\Big)\,||\nabla u(t)||_2^2 \\ & +\; \varepsilon\Big(pH(t) + \frac{p}{2}\,(g\circ\nabla u)(t) + \frac{p}{2}\,||u_t||_2^2 + \frac{p}{2}\Big(1 - \int_0^t g(s)\,ds\Big)\,||\nabla u(t)||_2^2\Big) \\ & -\; a\varepsilon \int_\Omega |u_t|^{m-2}\,u_t u(x,t)\,dx - \varepsilon\beta(g\circ\nabla u)(t) - \frac{\varepsilon}{4\beta}\int_0^t g(s)\,ds\,||\nabla u(t)||_2^2 \\ \ge\; & a(1-\alpha)H^{-\alpha}(t)\,||u_t||_m^m + \varepsilon\Big(1 + \frac{p}{2}\Big)\int_\Omega u_t^2(x,t)\,dx + \varepsilon pH(t) \\ & +\; \varepsilon\Big(\frac{p}{2} - \beta\Big)(g\circ\nabla u)(t) - a\varepsilon \int_\Omega |u_t|^{m-2}\,u_t u(x,t)\,dx \\ & +\; \varepsilon\left(\Big(\frac{p}{2} - 1\Big) - \Big(\frac{p}{2} - 1 + \frac{1}{4\beta}\Big)\int_0^t g(s)\,ds\right)||\nabla u(t)||_2^2\,, \end{aligned} \tag{2.16}$$

for some number β with $0 < \beta < p/2$. By recalling (2.2), the estimate (2.16) reduces to

$$\begin{aligned} L'(t) \;\ge\; & a(1-\alpha)H^{-\alpha}(t)\,||u_t||_m^m + \varepsilon\Big(1 + \frac{p}{2}\Big)\int_\Omega u_t^2(x,t)\,dx + \varepsilon pH(t) \\ & +\; \varepsilon a_1 (g\circ\nabla u)(t) + \varepsilon a_2\,||\nabla u(t)||_2^2 - a\varepsilon \int_\Omega |u_t|^{m-2} u_t u(x,t)\,dx \end{aligned} \tag{2.17}$$

where

$$a_1 \;=\; \frac{p}{2} - \beta \;>\; 0\,, \quad a_2 \;=\; \Big(\frac{p}{2} - 1\Big) - \Big(\frac{p}{2} - 1 + \frac{1}{4\beta}\Big)\int_0^\infty g(s)\,ds \;>\; 0\,.$$

To estimate the last term of (2.17), we use again Young's inequality

$$XY \;\le\; \frac{\delta^r}{r}\,X^r + \frac{\delta^{-q}}{q}\,Y^q\,, \quad X\,,Y \;\ge\; 0\,, \quad \text{for all} \quad \delta \;>\; 0\,, \quad \frac{1}{r} + \frac{1}{q} \;=\; 1$$

with $r = m$ and $q = m/(m-1)$. So we have

$$\int_\Omega |u_t|^{m-1}\,|u|\,dx \;\le\; \frac{\delta^m}{m}\,||u||_m^m + \frac{m-1}{m}\,\delta^{-m/(m-1)}\,||u_t||_m^m$$

which yields, by substitution in (2.17),

$$\begin{aligned} L'(t) \;\ge\; & a\left[(1-\alpha)H^{-\alpha}(t) - \frac{m-1}{m}\,\varepsilon\delta^{-m/(m-1)}\right]||u_t||_m^m + \varepsilon\Big(1 + \frac{p}{2}\Big)\int_\Omega u_t^2(x,t)\,dx \\ & +\; \varepsilon a_1 (g\circ\nabla u)(t) + \varepsilon a_2\,||\nabla u(t)||_2^2 + \varepsilon pH(t) - \varepsilon a\,\frac{\delta^m}{m}\,||u||_m^m\,, \ \text{for all} \ \delta \;>\; 0\,. \end{aligned} \tag{2.18}$$

Of course (2.18) remains valid even if δ is time dependant since the integral is taken over the x variable. Therefore by taking δ so that $\delta^{-m/(m-1)} = kH^{-\alpha}(t)$, for large k to be specified later, and substituting in (2.18) we arrive at

$$\begin{aligned} L'(t) \;\geq\; & a\left[(1-\alpha) - \frac{m-1}{m}\,\varepsilon k\right] H^{-\alpha}(t)\,||u_t||_m^m + \varepsilon\Big(\frac{p}{2}+1\Big)\int_\Omega u_t^2(x,t)\,dx \\ & +\; \varepsilon a_1 (g\circ\nabla u)(t) + \varepsilon a_2\,||\nabla u(t)||_2^2 + \varepsilon\left[pH(t) - \frac{k^{1-m}}{m}\,aH^{\alpha(m-1)}(t)\,||u||_m^m\right]. \end{aligned} \tag{2.19}$$

By exploiting (2.11) and the inequality $||u||_m^m \leq C\,||u||_p^m$, we obtain

$$H^{\alpha(m-1)}(t)\,||u||_m^m \;\leq\; \left(\frac{b}{p}\right)^{\alpha(m-1)} C\,||u||_p^{m+\alpha p(m-1)}\,,$$

hence (2.19) yields

$$\begin{aligned} L'(t) \;\geq\; & a\left[(1-\alpha) - \frac{m-1}{m}\,\varepsilon k\right] H^{-\alpha}(t)\,||u_t||_m^m \\ & +\; \varepsilon\Big(\frac{p}{2}+1\Big)\int_\Omega u_t^2(x,t)\,dx + \varepsilon a_1(g\circ\nabla u)(t) + \varepsilon a_2\,||\nabla u(t)||_2^2 \\ & +\; \varepsilon\left[pH(t) - \frac{k^{1-m}}{m}\,a\Big(\frac{b}{p}\Big)^{\alpha(m-1)} C\,||u||_p^{m+\alpha p(m-1)}\right]. \end{aligned} \tag{2.20}$$

We then use Corollary 2.6 and (2.13), for $s = m + \alpha p(m-1) \leq p$, to deduce from (2.20)

$$\begin{aligned} L'(t) \;\geq\; & a\left[(1-\alpha) - \frac{m-1}{m}\,\varepsilon k\right] H^{-\alpha}(t)\,||u_t||_m^m \\ & +\; \varepsilon\Big(\frac{p}{2}+1\Big)\int_\Omega u_t^2(x,t)\,dx + \varepsilon a_1(g\circ\nabla u)(t) + \varepsilon a_2\,||\nabla u(t)||_2^2 \\ & +\; \varepsilon\left[pH(t) - C_1 k^{1-m}\left\{-H(t) - ||u_t||_2^2 - (g\circ\nabla u)(t) + ||u||_p^p\right\}\right] \\ \geq\; & a\left[(1-\alpha) - \frac{m-1}{m}\,\varepsilon k\right] H^{-\alpha}(t)\,||u_t||_m^m \\ & +\; \varepsilon\Big(\frac{p}{2}+1+C_1k^{1-m}\Big)\,||u_t||_2^2 + \varepsilon\left(a_1 + C_1k^{1-m}\right)(g\circ\nabla u)(t) \\ & +\; \varepsilon a_2\,||\nabla u(t)||_2^2 + \varepsilon\big(p + C_1k^{1-m}\big)H(t) - \varepsilon C_1 k^{1-m}\,||u||_p^p \end{aligned} \tag{2.21}$$

where $C_1 = a\left(\frac{b}{p}\right)^{\alpha(m-1)} C/m$. By noting that

$$H(t) \;\geq\; \frac{b}{p}\,||u||_p^p - \frac{1}{2}\,||u_t||_2^2 - \frac{1}{2}\,||\nabla u||_2^2 - \frac{1}{2}\,(g\circ\nabla u)(t)$$

and writing $p = 2a_3 + (p - 2a_3)$, where $a_3 = \min\{a_1, a_2\}$, the estimate (2.21) yields

$$\begin{aligned} L'(t) \;\geq\; & a\left[(1-\alpha) - \frac{m-1}{m}\,\varepsilon k\right] H^{-\alpha}(t)\,||u_t||_m^m + \varepsilon\left(\frac{p}{2} + 1 + C_1k^{1-m} - a_3\right)||u_t||_2^2 \\ & +\; \varepsilon\big(a_1 + C_1k^{1-m} - a_3\big)\,(g\circ\nabla u)(t) + \varepsilon(a_2 - a_3)\,||\nabla u(t)||_2^2 \\ & +\; \varepsilon\big(p - 2a_3 + C_1k^{1-m}\big)H(t) + \varepsilon\left(\frac{2ba_3}{p} - C_1k^{1-m}\right)||u||_p^p\,. \end{aligned} \tag{2.22}$$

At this point, we choose k large enough so that (2.22) becomes

$$\begin{aligned} L'(t) \;\geq\; & a\left[(1-\alpha) - \frac{m-1}{m}\,\varepsilon k\right] H^{-\alpha}(t)\,||u_t||_m^m \\ & +\varepsilon\gamma\left[H(t) + ||u_t||_2^2 + ||u||_p^p + (g\circ\nabla u)(t)\right], \end{aligned} \tag{2.23}$$

where $\gamma > 0$ is the minimum of the coefficients of $H(t)$, $||u_t||_2^2$, $||u||_p^p$, and $(g \circ \nabla u)(t)$ in (2.23). Once k is fixed (hence γ), we pick ε small enough so that

$$(1-\alpha) - \varepsilon k(m-1)/m \ \geq \ 0$$

and

$$L(0) \ = \ H^{1-\alpha}(0) + \varepsilon \int_\Omega u_0 u_1(x)\, dx \ > \ 0\,.$$

Therefore (2.23) takes the form

$$L'(t) \ \geq \ \varepsilon\gamma\big[H(t) + ||u_t||_2^2 + ||u||_p^p + (g \circ \nabla u)(t)\big]\ . \tag{2.24}$$

Consequently we have

$$L(t) \ \geq \ L(0) \ > \ 0\,, \quad \text{for all} \quad t \ \geq \ 0\,.$$

We now estimate

$$\left|\int_\Omega u u_t(x,t)\, dx\right| \ \leq \ ||u||_2\, ||u_t||_2 \ \leq \ C\, ||u||_p\, ||u_t||_2$$

which implies

$$\left|\int_\Omega u u_t(x,t)\, dx\right|^{1/(1-\alpha)} \ \leq \ C\, ||u||_p^{1/(1-\alpha)}\, ||u_t||_2^{1/(1-\alpha)}\,.$$

Again Young's inequality gives us

$$\left|\int_\Omega u u_t(x,t)\, dx\right|^{1/(1-\alpha)} \ \leq \ C\left[||u||_p^{\mu/(1-\alpha)} + ||u_t||_2^{\theta/(1-\alpha)}\right]\,, \tag{2.25}$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\alpha)$, to get $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p$ by (2.14). Therefore (2.25) becomes

$$\left|\int_\Omega u u_t(x,t)\, dx\right|^{1/(1-\alpha)} \ \leq \ C\left[||u||_p^s + ||u_t||_2^2\right]\,,$$

where $s = 2/(1-2\alpha) \leq p$. By using Corollary 2.6 we obtain

$$\left|\int_\Omega u u_t(x,t)\, dx\right|^{1/(1-\alpha)} \ \leq \ C\left[H(t) + ||u||_p^p + ||u_t||_2^2 + (g \circ \nabla u)(t)\right], \ \text{for all} \ t \ \geq \ 0\,. \tag{2.26}$$

Therefore we have

$$\begin{aligned} L^{1/(1-\alpha)}(t) \ &= \ \left(H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t)\, dx\right)^{1/(1-\alpha)} \\ &\leq \ 2^{1/(1-\alpha)}\left(H(t) + \left|\int_\Omega u u_t(x,t)\, dx\right|^{1/(1-\alpha)}\right) \\ &\leq \ C\left[H(t) + ||u||_p^p + ||u_t||_2^2 + (g \circ \nabla u)(t)\right]\,, \quad \text{for all} \quad t \ \geq \ 0\,. \end{aligned} \tag{2.27}$$

By combining (2.24) and (2.27) we arrive

$$L'(t) \ \geq \ \Gamma L^{1/(1-\alpha)}(t)\,, \quad \text{for all} \quad t \ \geq \ 0\,, \tag{2.28}$$

where Γ is a positive constant depending only on $\varepsilon\gamma$ and C (the constant of Lemma 2.5). A simple integration of (2.28) over $(0,t)$ then yields

$$L^{\alpha/(1-\alpha)}(t) \ \geq \ \frac{1}{L^{\,-\alpha/(1-\alpha)}(0) - \Gamma t\alpha/(1-\alpha)}\,. \tag{2.29}$$

Therefore (2.29) shows that $L(t)$ blows up in a time given by the estimate (2.4) above. This completes the proof.

Remark 2.7 The estimate (2.4) shows that the larger $L(0)$ is, the quicker the blow up takes place.

3 Global existence

In this section we show that solution (1.4) is global if $m \geq p$.

Theorem 3.1 *Assume that* (2.3) *holds and* $2 \leq p \leq m$. *Assume further that*

$$m \;\leq\; \frac{2(n-1)}{n-2}\,, \quad n \;\geq\; 3\,. \tag{3.1}$$

Then for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *problem* (1.1) *has a unique solution*

$$u \in C\big([0,\infty); H_0^1(\Omega)\big)\,, \quad u_t \in C\big([0,\infty); L^2(\Omega)\big) \cap L^{m+1}(\Omega \times (0,\infty))\,. \tag{3.2}$$

Proof. Similar to [6], we set

$$F(t) \;=\; -H(t) + \frac{2b}{p}\,||u||_p^p \;=\; \frac{1}{2}\,||u_t||_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)||\nabla u||_2^2 + \frac{1}{2}\,(g \circ \nabla u)(t) + \frac{b}{p}\,||u||_p^p\,.$$

By differentiating $F(t)$ and using (2.10), we get

$$F'(t) \;=\; -a\int_\Omega |u_t|^m\,dx + \frac{1}{2}\,(g' \circ \nabla u)(t) - \frac{1}{2}\,g(t)\,||\nabla u(t)||^2 + 2b\int_\Omega |u|^{p-2}\,u u_t\,dx\,.$$

By using Young's inequality, we obtain

$$\begin{aligned} F'(t) \;&\leq\; -\,a\,||u_t||_m^m + \frac{1}{2}\,(g' \circ \nabla u)(t) - \frac{1}{2}\,g(t)\,||\nabla u(t)||^2 + \delta\,||u_t||_p^p + C_\delta\,||u||_p^p \\ &\leq\; -\,a\,||u_t||_m^m + \delta\,||u_t||_p^p + C_\delta\,||u||_p^p \end{aligned}$$

where $\delta > 0$ and C_δ is a constant depending on δ. By noting that $m \geq p$ we easily see that

$$F'(t) \;\leq\; -a\,||u_t||_m^m + C\delta\,||u_t||_m^p + C_\delta\,||u||_p^p$$

where $C = C(\Omega, p, m)$ is the embedding constant. At this point we distinguish two cases

1) Either $||u_t||_m^m > 1$ so we choose δ so small that $-a\,||u_t||_m^m + C\delta\,||u_t||_m^p \leq 0$; hence $F'(t) \leq C_\delta\,||u||_p^p$.

2) Or $||u_t||_m^m \leq 1$, in this case we have $F'(t) \leq C\delta + C_\delta\,||u||_p^p$.

Therefore in either case we have

$$F'(t) \;\leq\; c_1 + C_\delta\,||u||_p^p \;\leq\; c_1 + C_\delta F(t)\,. \tag{3.3}$$

A simple integration of (3.3) yields

$$F(t) \;\leq\; \left(F(0) + \frac{c_1}{C_\delta}\right)e^{C_\delta t}\,.$$

The last estimate together with the continuation principle completes our proof. □

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