MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2005; **28**:1819–1828
Published online 27 June 2005 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/mma.641
MOS subject classification: 35 L 45

On the decay of solutions for a class of quasilinear hyperbolic equations with non-linear damping and source terms

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Communicated by H. A. Levine

SUMMARY

In this paper, we consider the non-linear wave equation

$$u_{tt} - \Delta u_t - \operatorname{div}(|\nabla u|^m \nabla u) + a|u_t|^\alpha u_t = b|u|^p u$$

$a, b > 0$, associated with initial and Dirichlet boundary conditions. Under suitable conditions on α, m, and p, we give precise decay rates for the solution. In particular, we show that for $m = 0$, the decay is exponential. This work improves the result by Yang (*Math. Meth. Appl. Sci.* 2002; **25**:795–814). Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: non-linear damping; non-linear source; global existence; polynomial decay; exponential decay

1. INTRODUCTION

In this paper, we are concerned with the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u_t - \operatorname{div}(|\nabla u|^m \nabla u) + a|u_t|^\alpha u_t = b|u|^p u, & x \in \Omega, \ t > 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega, \ t > 0 \end{cases} \tag{1}$$

where a, b, α, m, $p > 0$ and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$, with a smooth boundary $\partial\Omega$.

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Contract/grant sponsor: King Fahd University of Petroleum and Minerals
Contract/grant sponsor: KFUPM; contract/grant number: MS/VISCO ELASTIC 270

Received 8 June 2004

Equation (1) appears in the models of non-linear viscoelasticity and it can also be considered as a system governing the longitudinal motion of a viscoelastic configuration obeying a non-linear Voight model (see References [1–4]).

In the absence of the strong damping $-\Delta u_t$, Equation (1) becomes

$$u_{tt} - \operatorname{div}(|\nabla u|^m \nabla u) + a|u_t|^\alpha u_t = b|u|^p u, \quad x \in \Omega, \; t > 0 \tag{2}$$

For $b=0$, it is well known that the damping term assures global existence and decay of the solution energy for arbitrary initial data (see References [5–7]). For $a=0$, the source term causes finite time blow up of solutions with negative initial energy if $p>m$ (see Reference [8]).

The interaction between the damping and the source terms was first considered by Levine [9,10] in the linear damping case ($\alpha = m = 2$). He showed that solutions with negative initial energy blow up in finite time. This result was later improved by Kalantarov and Ladyzhenskaya [11] to accommodate more situations. Georgiev and Todorova [12] extended Levine's result to the non-linear damping case ($\alpha > 2$). In their work, the authors considered (2) with $m = 2$ and introduced a method different than the one known as the concavity method. They determined suitable relations between α and p, for which there is global existence or alternatively finite time blow up. Precisely; they showed that solutions with negative energy continue to exist globally 'in time' if $\alpha \geqslant p$ and blow up in finite time if $p > \alpha$ and the initial energy is sufficiently negative. This result was later generalized to an abstract setting and to unbounded domains by Levine and Serrin [13] and Levine, *et al.* [14]. In these papers, the authors showed that no solution with negative energy can be extended on $[0, \infty)$ if $p > \alpha$ and proved several non-continuation theorems. This generalization allowed them also to apply their result to quasilinear situations ($m>2$), of which the problem in Reference [12] is a particular case. Vitillaro [15] extended these results to situations where the damping is non-linear and the solution has positive initial energy. Similar results have also been established by Todorova [16,17] for different Cauchy problems.

In Reference [4], Zhijian studied a similar problem to (1) and proved a blow up result under the condition $p > \max\{\alpha, m\}$ and the initial energy is sufficiently negative (see condition ii, Theorem 2.1 of Reference [4]). In fact this condition made it clear that there exists a certain relation between the blow-up time and $|\Omega|$ (see Remark 2 of Reference [4]). Messaoudi and Said-Houari [18] improved Zhijian result and showed that the blow up takes place for negative initial data only regardless of the size of Ω. In a recent work, Yang [1] also considered

$$\begin{cases} u_{tt} - \Delta u_t - \sum_{i=1}^{i=n} \dfrac{\partial}{\partial x_i} \sigma_i(u_{x_i}) + f(u_t) = g(u), \quad (x,t) \in \Omega \times (0, +\infty) \\ u(.,t)|_{\partial\Omega} = 0, \quad t \geqslant 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \end{cases} \tag{3}$$

where σ_i ($i = 1, \ldots, n$), f and g are non-linear function behaving like $|s|^m s$, $|s|^\alpha s$, and $|s|^p s$, respectively. He combined the Galerkin method with the potential well technique and proved a global existence result of weak solutions and a decay result of the form

$$\|u_t(t)\|_2^2 + \|\nabla u(t)\|_{m+2}^{m+2} + \|u(t)\|_{p+2}^{p+2} = O\left(\frac{t^{(2m+3)/(2m+4)}}{t+1}\right) \quad \text{as } t \to +\infty$$

In this work, we improve the result of Reference [1] by giving more precise decay rates. In particular, we show that for $m = 0$, the decay is exponential. Our technique of proof relies on the combination of the perturbed energy and the potential well methods.

For the sake of completeness we state, without proof, the global existence result of Reference [1]. For this aim, we set

$$\mathbb{W} = \{v \in W_0^{1,m+2}(\Omega) \backslash \|\nabla v\|_{m+2}^{m+2} - b\|v\|_{p+2}^{p+2} > 0\} \cup \{0\}$$

We will also need the Sobolev embedding of $W_0^{1,m+2}(\Omega)$ in $L^q(\Omega)$

$$\|u\|_q \leqslant C_* \|\nabla u\|_{m+2}, \quad q \leqslant \frac{n(m+2)}{n-(m+2)}, \quad m+2 < n \tag{4}$$

and

$$E_0 = \frac{1}{2}\int_\Omega u_1^2 \, dx + \frac{1}{m+2}\int_\Omega |\nabla u_0|^{m+2} \, dx - \frac{1}{p+2}\int_\Omega |u_0|^{p+2} \, dx$$

Proposition

Suppose that m, $\alpha \geqslant 0$, and $p > m$ such that

$$\max\{\alpha+2, p+2\} \leqslant \frac{n(m+2)}{n-(m+2)}, \quad m+2 < n \tag{5}$$

and let $(u_0, u_1) \in \mathbb{W} \times L^2(\Omega)$ be given and satisfying

$$\beta = bC_*^{p+2}\left(\frac{(m+2)(p+2)}{p-m}E_0\right)^{(p-m)/(m+2)} < 1 \tag{6}$$

Then problem (1) has a unique global solution:

$$\begin{aligned} u &\in L^\infty([0,\infty); W_0^{1,m+2}(\Omega)) \\ u_t &\in L^\infty([0,\infty); L^2(\Omega)) \cap L^{\alpha+2}(\Omega \times (0,T)) \cap L^2([0,\infty); H_0^1(\Omega)) \end{aligned} \tag{7}$$

2. MAIN RESULT

In order to state and prove our main result we first introduce the following functionals:

$$\begin{aligned} I(t) &= I(u(t)) = \|\nabla u(t)\|_{m+2}^{m+2} - b\|u(t)\|_{p+2}^{p+2} \\ J(t) &= J(u(t)) = \frac{1}{m+2}\|\nabla u(t)\|_{m+2}^{m+2} - \frac{b}{p+2}\|u(t)\|_{p+2}^{p+2} \\ E(t) &= E(u(t), u_t(t)) = J(t) + \tfrac{1}{2}\|u_t(t)\|_2^2 \end{aligned} \tag{8}$$

where we are using $w(t)$ instead of $w(.,t)$.

Remark 2.1

By multiplying Equation (1) by u_t, integrating over Ω, and using integration by parts, we get

$$E'(t) = -(a\|u_t(t)\|_{\alpha+2}^{\alpha+2} + \|\nabla u_t(t)\|_2^2) \leqslant 0 \tag{9}$$

for almost each t. Therefore,

$$E(t) \leqslant E_0 \quad \forall t \geqslant 0 \tag{10}$$

Lemma 2.1
Suppose that m, $\alpha \geqslant 0$, and $p > m$ such that (5) holds and let $(u_0, u_1) \in \mathbb{W} \times L^2(\Omega)$ be given and satisfying (6). Then $u(t) \in \mathbb{W}$, for each $t \geqslant 0$.

Proof
Since $I(u_0) > 0$ then there exists $T_m > 0$ such that $I(u(t)) \geqslant 0$ for all $t \in [0, T_m)$. This implies

$$\begin{aligned} J(t) &= \frac{1}{m+2}\|\nabla u(t)\|_{m+2}^{m+2} - \frac{b}{p+2}\|u(t)\|_{p+2}^{p+2} \\ &= \frac{p-m}{(m+2)(p+2)}\|\nabla u(t)\|_{m+2}^{m+2} + \frac{1}{p+2}I(u(t)) \\ &\geqslant \frac{p-m}{(m+2)(p+2)}\|\nabla u(t)\|_{m+2}^{m+2} \quad \forall t \in [0, T_m) \end{aligned} \tag{11}$$

hence

$$\begin{aligned} \|\nabla u(t)\|_{m+2}^{m+2} &\leqslant \frac{(m+2)(p+2)}{p-m}J(t) \leqslant \frac{(m+2)(p+2)}{p-m}E(t) \\ &\leqslant \frac{(m+2)(p+2)}{p-m}E_0 \quad \forall t \in [0, T_m) \end{aligned} \tag{12}$$

By exploiting (4)–(6) and (12), we easily arrive at

$$\begin{aligned} b\|u(t)\|_{p+2}^{p+2} &\leqslant bC_*^{p+2}\|\nabla u(t)\|_{m+2}^{p+2} = bC_*^{p+2}\|\nabla u(t)\|_{m+2}^{p-m}\|\nabla u(t)\|_{m+2}^{m+2} \\ &\leqslant bC_*^{p+2}\left(\frac{(m+2)(p+2)}{p-m}E_0\right)^{(p-m)/(m+2)}\|\nabla u(t)\|_{m+2}^{m+2} \\ &= \beta\|\nabla u(t)\|_{m+2}^{m+2} < \|\nabla u(t)\|_{m+2}^{m+2} \quad \forall t \in [0, T_m) \end{aligned} \tag{13}$$

hence

$$\|\nabla u(t)\|_{m+2}^{m+2} - b\|u(t)\|_{p+2}^{p+2} > 0 \quad \forall t \in [0, T_m)$$

This shows that $u(t) \in \mathbb{W}, \forall t \in [0, T_m)$. By noting that

$$\lim_{t \to T_m} bC_*^p\left(\frac{2p}{p-2}E(u(t), u_t(t))\right)^{(p-2)/2} \leqslant \beta$$

we easily repeat steps (11)–(13) to extend T_m to $2T_m$. By continuing the procedure, the assertion of the lemma is proved. □

Lemma 2.2
Suppose that (5) and (6) hold. Then for $\alpha \geqslant m$, we have

$$\|u(t)\|_{\alpha+2}^{\alpha+2} \leqslant CE(t) \tag{14}$$

for some constant C depending on α, m, p, and E_0 only.

Proof
If $\alpha = m$ then (14) is trivial by virtue of (11) and (13).
If $\alpha > m$ then

$$\begin{aligned}\|u(t)\|_{\alpha+2}^{\alpha+2} &\leqslant C_*^{\alpha+2}\|\nabla u(t)\|_{m+2}^{\alpha+2} \leqslant C_*^{\alpha+2}\|\nabla u(t)\|_{m+2}^{m+2}\|\nabla u(t)\|_{m+2}^{\alpha-m}\\ &\leqslant C_*^{\alpha+2}\left(\frac{(m+2)(p+2)}{p-m}E_0\right)^{(\alpha-m)/(m+2)}\frac{(m+2)(p+2)}{p-m}E(t)\end{aligned} \tag{15}$$

again holds by virtue of (4)–(6) and (12). Therefore, (14) is established. □

Lemma 2.3
For Ω (bounded or unbounded), if $v \in L^q(\Omega) \cap L^r(\Omega)$, $q < r$ then $v \in L^s(\Omega)$, $q \leqslant s \leqslant r$. Moreover, there exist two constants c_1 and c_2 depending on q and r only such that

$$\|v\|_s^s \leqslant c_1\|v\|_q^q + c_2\|v\|_r^r \tag{16}$$

Proof
Since $q \leqslant s \leqslant r$ then $s = tq + (1-t)r$, $0 \leqslant t \leqslant 1$. So by Young's inequality we have $|v|^s \leqslant c_1|v|^q + c_2|v|^r$. Integrating over Ω, (16) is established. □

Lemma 2.4
Under conditions (5), (6) and if $\alpha \geqslant m/(m+2)$, the solution of (1) satisfies

$$\int_\Omega |u_t|^{\alpha+1}|u|\,\mathrm{d}x \leqslant \delta CE(t) + c(\delta)\{\|u_t\|_{\alpha+2}^{\alpha+2} + \|\nabla u_t\|_2^2\} \tag{17}$$

where C is the constant in (14), δ is any positive constant, and $c(\delta)$ is a constant depending on δ, α, m, and p only.

Proof

(1) if $\alpha \geqslant m$ then by Young's inequality and (14) we have for any $\delta > 0$,

$$\begin{aligned}\int_\Omega |u_t|^{\alpha+1}|u|\,\mathrm{d}x &\leqslant \delta\int_\Omega |u|^{\alpha+2}\,\mathrm{d}x + c(\delta)\int_\Omega |u_t|^{\alpha+2}\,\mathrm{d}x\\ &\leqslant \delta CE(t) + c(\delta)\{\|u_t\|_{\alpha+2}^{\alpha+2} + \|\nabla u_t\|_2^2\}\end{aligned} \tag{18}$$

(2) if $m/(m+2) \leqslant \alpha < m$ then we have

$$\int_\Omega |u_t|^{\alpha+1}|u|\,\mathrm{d}x \leqslant \delta\int_\Omega |u|^{m+2}\,\mathrm{d}x + c(\delta)\int_\Omega |u_t|^{(\alpha+1)(m+2)/(m+1)}\,\mathrm{d}x \tag{19}$$

By noting that $2 \leqslant (\alpha+1)(m+2)/(m+1) < \alpha+2$, we easily see, from (16), that

$$\int_\Omega |u_t|^{(\alpha+1)(m+2)/(m+1)}\,dx \leqslant c_1 \int_\Omega |u_t|^2\,dx + c_2 \int_\Omega |u_t|^{(\alpha+2)}\,dx$$

$$\leqslant c_3 \int_\Omega |\nabla u_t|^2\,dx + c_2 \int_\Omega |u_t|^{(\alpha+2)}\,dx \tag{20}$$

Combining (18) and (20) and recalling (12) the lemma is proved. □

Theorem 2.5

Suppose that $\alpha \geqslant m/(m+2) \geqslant 0$, and $p > m$ such that (5) is satisfied and let $(u_0, u_1) \in \mathbb{W} \times L^2(\Omega)$ be given and satisfying (6). Then there exist positive constants K and k such that, for all $t \geqslant 0$, the global solution satisfies

$$\begin{aligned} E(t) &\leqslant K e^{-kt}, \quad m = 0 \\ E(t) &\leqslant (kt + K)^{-2/m}, \quad m > 0 \end{aligned} \tag{21}$$

Proof

We define

$$F(t) := E(t) + \varepsilon \int_\Omega \left(u(t)u_t(t) + \frac{1}{2}|\nabla u(t)|^2 \right) dx \tag{22}$$

and note that for ε small enough, there exist two positive constants α_1 and α_2 such that

$$\alpha_1 E(t) \leqslant F(t) \leqslant \alpha_2 (E(t))^{2/(m+2)} \tag{23}$$

In fact

$$\begin{aligned} F(t) &\leqslant E(t) + \frac{\varepsilon}{2}\int_\Omega |u_t(t)|^2\,dx + \frac{\varepsilon}{2}\int_\Omega |u(t)|^2\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla u(t)|^2\,dx \\ &\leqslant \left(1 + \frac{\varepsilon}{2}\right) E(t) + \varepsilon C(\Omega) \|\nabla u\|_{m+2}^2 \\ &\leqslant \left(1 + \frac{\varepsilon}{2}\right) E(t) + \varepsilon C(\Omega)(E(t))^{2/(m+2)} \\ &\leqslant \left(\left(1 + \frac{\varepsilon}{2}\right) E^{m/(m+2)}(t) + \varepsilon C(\Omega) \right) (E(t))^{2/(m+2)} \\ &\leqslant \left(\left(1 + \frac{\varepsilon}{2}\right) (E_0)^{m/(m+2)} + \varepsilon C(\Omega) \right) (E(t))^{2/(m+2)} = \alpha_2 (E(t))^{2/(m+2)} \end{aligned}$$

and

$$F(t) \geqslant E(t) - \varepsilon \left\{ \frac{1}{4\gamma} \int_\Omega |u_t(t)|^2\,dx + \gamma \int_\Omega |u(t)|^2\,dx \right\} + \frac{\varepsilon}{2}\int_\Omega |\nabla u(t)|^2\,dx$$

$$\geqslant E(t) - \frac{\varepsilon}{4\gamma}\int_\Omega |u_t(t)|^2\,dx + \varepsilon \left\{ \frac{1}{2} - C(\Omega)\gamma \right\} \int_\Omega |\nabla u(t)|^2\,dx \tag{24}$$

By choosing γ small enough (24) yields

$$\begin{aligned} F(t) &\geqslant E(t) - \frac{\varepsilon}{4\gamma}\int_\Omega |u_t(t)|^2\,\mathrm{d}x \\ &\geqslant J(t) + \left(1 - \frac{\varepsilon}{4\gamma}\right)\int_\Omega |u_t(t)|^2\,\mathrm{d}x \end{aligned}$$

We then pick $\varepsilon > 0$ so small that

$$F(t) \geqslant J(t) + \alpha_1 \int_\Omega |u_t(t)|^2\,\mathrm{d}x \geqslant \alpha_1 E(t)$$

We now differentiate (22) and use Equation (1), (9) and Poincaré's inequality, to obtain

$$\begin{aligned} F'(t) &= -(a\|u_t(t)\|_{m+2}^{m+2} + \|\nabla u_t(t)\|_2^2) + \varepsilon\int_\Omega [u_t^2(t) - |\nabla u(t)|^{m+2}]\,\mathrm{d}x \\ &\quad - a\varepsilon\int_\Omega |u_t(t)|^\alpha u_t(t)u(t)\,\mathrm{d}x + \varepsilon b\int_\Omega |u(t)|^p\,\mathrm{d}x \\ &\leqslant -a\int_\Omega |u_t(t)|^{m+2}\,\mathrm{d}x - [1-\varepsilon C(\Omega)]\|\nabla u_t(t)\|_2^2 - \varepsilon\int_\Omega |\nabla u(t)|^{m+2}\,\mathrm{d}x \\ &\quad + a\varepsilon\int_\Omega |u_t(t)|^{\alpha+1}|u(t)|\,\mathrm{d}x + \varepsilon b\int_\Omega |u(t)|^p\,\mathrm{d}x \end{aligned} \tag{25}$$

We then use (8) and (13) to get

$$\begin{aligned} b\int_\Omega |u(t)|^p\,\mathrm{d}x &= \lambda b\int_\Omega |u(t)|^p\,\mathrm{d}x + (1-\lambda)b\int_\Omega |u(t)|^p\,\mathrm{d}x \\ &\leqslant \lambda\left(\frac{p+2}{2}\int_\Omega u_t^2(t)\,\mathrm{d}x + \frac{p+2}{m+2}\int_\Omega |\nabla u(t)|^{m+2}\,\mathrm{d}x - (p+2)E(t)\right) \\ &\quad + (1-\lambda)\beta\int_\Omega |\nabla u(t)|^{m+2}\,\mathrm{d}x, \quad 0<\lambda<1 \end{aligned} \tag{26}$$

By using Lemma 2.4 and (26), estimate (25) becomes

$$\begin{aligned} F'(t) &\leqslant -a\int_\Omega |u_t(t)|^{m+2}\,\mathrm{d}x - [1-\varepsilon C(\Omega)]\|\nabla u_t(t)\|_2^2 - \varepsilon\int_\Omega |\nabla u(t)|^{m+2}\,\mathrm{d}x \\ &\quad + \varepsilon\lambda\left(\frac{p+2}{2}\int_\Omega u_t^2(t)\,\mathrm{d}x + \frac{p+2}{m+2}\int_\Omega |\nabla u(t)|^{m+2}\,\mathrm{d}x - (p+2)E(t)\right) \\ &\quad + \varepsilon(1-\lambda)\beta\int_\Omega |\nabla u(t)|^{m+2}\,\mathrm{d}x + a\varepsilon\delta CE(t) + a\varepsilon c(\delta)\{\|u_t\|_{\alpha+2}^{\alpha+2} + \|\nabla u_t\|_2^2\} \end{aligned}$$

$$\leqslant -a[1-\varepsilon c(\delta)]\|u_t\|_{\alpha+2}^{\alpha+2} - \left(1-\varepsilon\left[C(\Omega)+ac(\delta)+\lambda\frac{p+2}{2}C(\Omega)\right]\right)\|\nabla u_t\|_2^2$$
$$-\varepsilon[\lambda(p+2)-a\delta]E(t)+\varepsilon\left((1-\lambda)\beta+\lambda\frac{p+2}{m+2}-1\right)\int_\Omega|\nabla u(t)|^{m+2}\,\mathrm{d}x \qquad (27)$$

By setting $\eta = 1-\beta$, (27) yields

$$F'(t) \leqslant -\left(1-\varepsilon\left[C(\Omega)+ac(\delta)+\lambda\frac{p+2}{2}C(\Omega)\right]\right)\|\nabla u_t\|_2^2$$
$$-a[1-\varepsilon c(\delta)]\|u_t\|_{\alpha+2}^{\alpha+2}-\varepsilon[\lambda(p+2)-a\delta]E(t)$$
$$+\varepsilon\left(\frac{p-m}{m+2}\lambda-\eta(1-\lambda)\right)\int_\Omega|\nabla u(t)|^{m+2}\,\mathrm{d}x \qquad (28)$$

By using (12) and choosing λ close to 1 so that $(p-m)/(m+2)\lambda-\eta(1-\lambda)\geqslant 0$, we arrive at

$$F'(t) \leqslant -a[1-\varepsilon c(\delta)]\|u_t\|_{\alpha+2}^{\alpha+2} - \left(1-\varepsilon\left[C(\Omega)+ac(\delta)+\lambda\frac{p+2}{2}C(\Omega)\right]\right)\|\nabla u_t\|_2^2$$
$$-\varepsilon[\lambda(p+2)-a\delta]E(t)+\varepsilon\left((p+2)\lambda-\frac{(m+2)(p+2)}{p-m}\eta(1-\lambda)\right)E(t)$$
$$\leqslant -a[1-\varepsilon c(\delta)]\|u_t\|_{\alpha+2}^{\alpha+2} - \left(1-\varepsilon\left[C(\Omega)+ac(\delta)+\lambda\frac{p+2}{2}C(\Omega)\right]\right)\|\nabla u_t\|_2^2$$
$$-\varepsilon\left(\eta\frac{(m+2)(p+2)}{p-m}(1-\lambda)-a\delta\right)E(t) \qquad (29)$$

At this point we choose δ so small that $\eta[((m+2)(p+2))/(p-m)](1-\lambda)-a\delta>0$. Once δ is chosen we then pick ε so small that

$$1-\varepsilon c(\delta)\geqslant 0,\quad 1-\varepsilon\left[C(\Omega)+ac(\delta)+\lambda\,\frac{p+2}{2}\,C(\Omega)\right]\geqslant 0$$

and (23) remains valid. Consequently, (29) yields

$$F'(t)\leqslant -\varepsilon\left(\eta\frac{(m+2)(p+2)}{p-m}(1-\lambda)-a\delta\right)E(t)$$
$$\leqslant -\frac{\varepsilon}{(\alpha_2)^{(m+2)/2}}\left(\eta\frac{(m+2)(p+2)}{p-m}(1-\lambda)-a\delta\right)F^{(m+2)/2}(t) \qquad (30)$$

We distinguish two cases.

(i) $m = 0$, then a simple integration of (30) leads to

$$E(t) \leqslant F(t) \leqslant F(0)\mathrm{e}^{-kt} \quad \forall t \geqslant 0 \tag{31}$$

where

$$k = \frac{\varepsilon}{(\alpha_2)^{(m+2)/2}} \left(\eta \frac{(m+2)(p+2)}{p-m}(1-\lambda) - a\delta \right)$$

(ii) $m > 0$, again a simple integration of (30) gives

$$E(t) \leqslant F(t) \leqslant (kt + F^{-m/2}(0))^{-2/m} \tag{32}$$

where

$$k = \frac{\varepsilon m}{2(\alpha_2)^{(m+2)/2}} \left(\eta \frac{(m+2)(p+2)}{p-m}(1-\lambda) - a\delta \right)$$

This completes the proof. □

Remark 2.2
By using (8), (12), (13) and (21), we easily obtain, for all $t \geqslant 0$,

$$\begin{aligned} \|u_t(t)\|_2^2 + \|\nabla u(t)\|_{m+2}^{m+2} + \|u(t)\|_{p+2}^{p+2} &\leqslant C\mathrm{e}^{-kt/2}, \quad m = 0 \\ \|u_t(t)\|_2^2 + \|\nabla u(t)\|_{m+2}^{m+2} + \|u(t)\|_{p+2}^{p+2} &\leqslant C(t+1)^{-2/(m+2)m}, \quad m > 0 \end{aligned} \tag{33}$$

Remark 2.3
Theorem 2.5 remains valid if $\alpha = 0$. In this case, we take

$$F(t) := E(t) + \varepsilon \int_\Omega \left(u(t)u_t(t) + \frac{1}{2}|\nabla u(t)|^2 + \frac{a}{2}|u(t)|^2 \right) \mathrm{d}x \tag{34}$$

and the same proof works.

Remark 2.4
Note that, for the case $m > 0$, Theorem 2.5 gives no information about the rate of decay if $0 < \alpha < m/(m+2)$.

ACKNOWLEDGEMENTS

The authors would like to express their sincere thanks to King Fahd University of Petroleum and Minerals for its support. This work has been funded by KFUPM under Project # MS/VISCO ELASTIC 270.

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