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Exponential stability in one-dimensional non-linear thermoelasticity with second sound

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SUMMARY

In this paper, we consider a one-dimensional non-linear system of thermoelasticity with second sound. We establish an exponential decay result for solutions with small 'enough' initial data. This work extends the result of Racke (*Math. Methods Appl. Sci.* 2002; **25**:409–441) to a more general situation.

KEY WORDS: non-linear; thermoelasticity; second sound; second law of thermodynamics; exponential decay

1. INTRODUCTION

For a one-dimensional homogeneous body occupying, in its reference configuration, an interval I the laws of balance of momentum, balance of energy, and growth of entropy have the forms

$$\rho u_{tt} = \sigma_x + b \tag{1}$$

$$e_t + q_x = \sigma \varepsilon_t + r \tag{2}$$

$$\eta_t \geqslant \frac{r}{\theta} - \left(\frac{q}{\theta}\right)_x \tag{3}$$

where the displacement u, the strain $\varepsilon = u_x$, the stress σ, the 'absolute' temperature θ, the heat flux q, the internal energy e, the body force b, and the external heat supply r are all functions

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of (x,t) $(t \geqslant 0,\ x \in I=(0,1))$. Moreover, the strain and the temperature are required to satisfy $\varepsilon > -1$ and $\theta > 0$. We then define the free energy by

$$\psi = e - \theta\eta \tag{4}$$

For thermoelasticity with second sound, the constitutive relations are

$$\psi = \hat{\psi}(\varepsilon,\theta,q), \quad \eta = \hat{\eta}(\varepsilon,\theta,q), \quad \sigma = \hat{\sigma}(\varepsilon,\theta,q), \quad e = \hat{e}(\varepsilon,\theta,q) \tag{5}$$

and the heat conduction is given by Cattaneo's law instead of Fourier's law

$$\tau(\varepsilon,\theta)q_t + q = -k(\varepsilon,\theta)\theta_x \tag{6}$$

where $\hat{\psi}, \hat{\eta}, \hat{\sigma}, \hat{e}, \tau$, and k are smooth functions. We note here that τ is the thermal relaxation time and k is the thermal conductivity.

Using the second law of thermodynamics (see References [1–3]), one can show that

$$\hat{\psi}(\varepsilon,\theta,q) = \psi^0(\varepsilon,\theta) + \frac{1}{2}\chi(\varepsilon,\theta)q^2, \quad \chi(\varepsilon,\theta) = \frac{\tau(\varepsilon,\theta)}{\theta k(\varepsilon,\theta)}$$

$$\hat{\sigma}(\varepsilon,\theta,q) = \hat{\psi}_\varepsilon(\varepsilon,\theta,q), \quad \hat{\eta}(\varepsilon,\theta,q) = -\hat{\psi}_\theta(\varepsilon,\theta,q) \tag{7}$$

It then follows from (5) and (7) that

$$\hat{e}(\varepsilon,\theta,q) = \hat{\psi}(\varepsilon,\theta,q) - \theta\hat{\psi}_\theta(\varepsilon,\theta,q) \tag{8}$$

which gives, in turn,

$$\hat{e}_\theta = -\theta\hat{\psi}_{\theta\theta}, \quad \frac{\hat{\sigma} - \hat{e}_\varepsilon}{\hat{\sigma}_\theta} = \theta \tag{9}$$

In the absence of the body force b and the external heat supply r, assuming that the material density ρ equal to one, and taking in consideration (7)–(9), Equations (1), (2), together with Cattaneo's law (6) take the form

$$u_{tt} - a(u_x,\theta,q)u_{xx} + b(u_x,\theta,q)\theta_x = \alpha_1(u_x,\theta)qq_x \tag{10}$$

$$\theta_t + g(u_x,\theta,q)q_x + d(u_x,\theta,q)u_{tx} = \alpha_2(u_x,\theta)qq_t \tag{11}$$

$$\tau(u_x,\theta)q_t + q + k(u_x,\theta)\theta_x = 0 \tag{12}$$

where

$$a = \hat{\sigma}_\varepsilon, \quad b = -\hat{\sigma}_\theta, \quad \alpha_1 = \chi_\varepsilon, \quad g = \frac{-1}{\theta\hat{\psi}_{\theta\theta}}, \quad d = \frac{\hat{\sigma}_\theta}{\hat{\psi}_{\theta\theta}}, \quad \alpha_2 = \frac{\chi - \theta\chi_\theta}{\theta\hat{\psi}_{\theta\theta}}$$

Tarabek [3] treated problems related to (10)–(12) in both bounded and unbounded situations and established global existence results for small initial data. He also showed that these 'classical' solutions tend to equilibrium as t tends to infinity; however, no rate of decay has been discussed. In his work, Tarabek used the usual energy argument and exploited some relations from the second law of thermodynamics[§] to overcome the difficulty arising from the lack of Poincare's inequality in the unbounded domains. Saouli [4] used the non-linear semigroup theory to prove a local existence result for a system similar to the one considered by Tarabek.

Concerning the asymptotic behaviour, Racke [5] discussed lately (10)–(12) and established exponential decay results for several linear and non-linear initial boundary value problems. In particular he studied system (10)–(12), for a rigidly clamped medium with temperature hold constant on the boundary, i.e.

$$u(t,0)=u(t,1)=\theta(t,0)=\theta(t,1)=\bar{\theta}, \quad t\geqslant 0$$

and showed that, for small enough initial data and for $\alpha_1=\alpha_2=0$, classical solutions decay exponentially to the equilibrium state. It is interesting to observe that taking $\alpha_1=\alpha_2=0$ makes $\chi(\varepsilon,\theta)=c_0\theta$, $c_0>0$, and consequently $\tau=c_0\theta k$, by virtue of (7). Although the dissipative effects of heat conduction induced by Cattaneo's law are usually weaker than those induced by Fourier's law, a global existence as well as exponential decay results for small initial data have been established. For a discussion in this direction, see Reference [5].

For the multi-dimensional case ($n=2,3$), Racke [6] established an existence result for the following n-dimensional problem

$$\begin{gathered} u_{tt}-\mu\Delta u-(\mu+\lambda)\nabla\operatorname{div} u+\beta\nabla\theta=0 \\ \theta_t+\gamma\operatorname{div} q+\delta\operatorname{div} u_t=0 \\ \tau q_t+q+k\nabla\theta=0, \quad x\in\Omega, \ t>0 \\ u(.,0)=u_0, \quad u_t(.,0)=u_1, \quad \theta(.,0)=\theta_0, \quad q(.,0)=q_0, \quad x\in\Omega \\ u=\theta=0, \quad x\in\partial\Omega, \quad t\geqslant 0 \end{gathered} \tag{13}$$

where Ω is a bounded domain of $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$, $u=u(x,t)$, $q=q(x,t)\in\mathbb{R}^n$, and $\mu,\lambda,\beta,\gamma,\delta,\tau,k$ are positive constants, where μ,λ are Lame moduli and τ is the relaxation time, a small parameter compared to the others. In particular if $\tau=0$, (13) reduces to the system of classical thermoelasticity, in which the heat flux is given by Fourier's law instead of Cattaneo's law. He also proved, under the conditions $\operatorname{rot} u=\operatorname{rot} q=0$, an exponential decay result for (13). This result applies automatically to the radially symmetric solution, since it is only a special case. Messaoudi [7] investigated the situation where a non-linear source term is competing with the damping caused by the heat conduction and established a local existence result. He also showed that solutions with negative energy blow up in finite time. His work generalized an earlier one in References [8,9] to thermoelasticity with second sound.

For the classical thermoelasticity ($\tau=0$), results concerning existence, blow up, and asymptotic behaviours of smooth, as well as weak, solutions have been established by several authors over the past two decades. See in this regard References [8–23]. In particular, we mention here Slemrod [23] who studied a one-dimensional non-linear system and proved the global existence and the asymptotic stability of solution for boundary conditions of the form $u_x=\theta=0$

[§]Relations from thermodynamics have been also used by Hrusa and Tarabek [11] to prove a global existence for the Cauchy problem to a classical thermoelasticity system and then by Hrusa and Messaoudi [12] to establish a blow up result for a thermoelastic system.

or $u=\theta_x=0$. For the global existence and asymptotic behaviour in the case of Dirichlet–Dirichlet boundary conditions ($u=\theta=0$), the problem remained open for a long time. In 1993, Racke *et al.* [21] then solved it by using a tricky way¶ to deal with the boundary terms. In fact this tricky way is due to Munoz Rivera [16] who considered the Dirichlet-Dirichlet boundary value problem in linear thermoelasticity and proved the exponential decay of solutions

In this paper, we consider (10)–(12), for a rigidly clamped medium with temperature hold constant at the boundary, and show that a similar argument to the one in Reference [5] is still valid to prove the exponential decay for classical solutions with small initial data. This work is organized as follows. In Section 2, we state the problem and in Section 3, we prove our main result.

2. STATEMENT OF THE PROBLEM

We consider the problem

$$u_{tt} - au_{xx} + b\theta_x = \alpha_1 q q_x \tag{14}$$

$$\theta_t + gq_x + du_{tx} = \alpha_2 q q_t \tag{15}$$

$$\tau q_t + q + k\theta_x = 0 \tag{16}$$

$$u(0,.) = u_0, \quad u_t(0,.) = u_1, \quad \theta(0,.) = \theta_0, \quad q(0,.) = q_0 \tag{17}$$

$$u(t,0) = u(t,1) = \theta(t,0) = \theta(t,1) = 0 \tag{18}$$

where

$$a = a(u_x, \theta, q), \quad b = b(u_x, \theta, q), \quad g = g(u_x, \theta, q), \quad d = d(u_x, \theta, q)$$

$$\tau = \tau(u_x, \theta), \quad k = k(u_x, \theta), \quad \alpha_1 = \alpha_1(u_x, \theta), \quad \alpha_2 = \alpha_2(u_x, \theta)$$

We assume that there exists a positive constant $\beta > 0$ such that

$$\beta \leqslant a(u_x, \theta, q), \quad \beta \leqslant g(u_x, \theta, q), \quad \beta \leqslant k(u_x, \theta), \quad \beta \leqslant \tau(u_x, \theta) \tag{19}$$

$$d(u_x, \theta, q) \neq 0, \quad b(u_x, \theta, q) \neq 0 \tag{20}$$

In order to make this paper self contained we state, without proof, a local existence result. The proof can be established by a classical energy argument [24]. For this purpose we set

$$u_2 = a(u_{0x}, \theta_0, q_0)u_{0xx} - b(u_{0x}, \theta_0, q_0)\theta_{0x} + \alpha_1(u_{0x}, \theta_0)q_0 q_{0x}$$

¶This way of dealing with boundary terms was also used by Racke in Reference [5] and it will be used in this work.

$$\theta_1 = -g(u_{0x}, \theta_0, q_0)q_{0x} - d(u_{0x}, \theta_0, q_0)u_{1x} + \alpha_2(u_{0x}, \theta_0)q_0 q_1$$

$$q_1 = \frac{-1}{\tau(u_{0x}, \theta_0)}\, q_0 - \frac{k(u_x, \theta)}{\tau(u_{0x}, \theta_0)}\, \theta_{0x}$$

Proposition
Assume that $a, b, \alpha_1, g, d, \alpha_2, \tau, k$ are C^3 functions satisfying (19) and (20). Then for any initial data

$$u_0 \in H^3(I) \cap H_0^1(I), \quad u_1, \theta_0 \in H^2(I) \cap H_0^1(I), \quad q_0 \in H^2(I)$$
$$u_2 \in H_0^1(I), \quad \theta_1 \in H_0^1(I), \quad \theta_0 > 0$$

problem (14)–(18) has a unique local solution (u, θ, q), on a maximal time interval $[0, T)$, satisfying

$$u \in \bigcap_{m=0}^{2} C^m([0,T), H^{3-m}(I) \cap H_0^1(I)), \quad \partial_t^3 u \in C([0,T), L^2(I))$$

$$\theta \in \bigcap_{m=0}^{1} C^m([0,T), H^{2-m}(I) \cap H_0^1(I)), \quad \partial_t^2 \theta \in C([0,T), L^2(I))$$

$$q \in \bigcap_{m=0}^{1} C^m([0,T), H^{2-m}(I) \cap H_0^1(I)), \quad \partial_t^2 q \in C([0,T), L^2(I))$$

To state our main result, we denote by

$$\begin{aligned}\Lambda(t) = \int_0^1 (&u_{ttt}^2 + u_{xxx}^2 + u_{ttx}^2 + u_{xxt}^2 + u_{tt}^2 + u_{xx}^2 + u_{tx}^2 + u_t^2 + u_x^2 \\ &+ \theta_{tt}^2 + \theta_{xx}^2 + \theta_{xt}^2 + \theta_t^2 + \theta_x^2 + \theta^2 + q_{tt}^2 + q_{xx}^2 + q_{xt}^2 + q_x^2 \\ &+ q_t^2 + q^2)\, \mathrm{d}x\end{aligned} \tag{21}$$

$$\alpha(t) = \sup_{0 \leqslant x \leqslant 1} \begin{pmatrix} |\theta| + |\theta_x| + |\theta_t| + |q| + |q_t| + |q_x| + |u_t| + |u_x| \\ + |u_{tx}| + |u_{xx}| + |u_{tt}| + |u| \end{pmatrix} \tag{22}$$

and

$$E(t) = E_1(t) + E_2(t) + E_3(t) \tag{23}$$

where

$$E_1(t) = \frac{1}{2} \int_0^1 [kdu_t^2 + kdau_x^2 + kb\theta^2 + bg\tau q^2](t,x)\, \mathrm{d}x = E_1(t, u, \theta, q) \tag{24}$$

and

$$E_2(t) = E_1(t, u_t, \theta_t, q_t), \quad E_3(t) = E_1(t, u_{tt}, \theta_{tt}, q_{tt})$$

Theorem
Assume that $a, b, \alpha_1, g, d, \alpha_2, \tau, k$ are C_b^3 functions satisfying (19) and (20). Then there exists a small positive constant δ such that if

$$\Lambda_0 = \|u_0\|_{H^3}^2 + \|u_1\|_{H^2}^2 + \|\theta_0\|_{H^2}^2 + \|q_0\|_{H^2}^2 < \delta \tag{25}$$

the solution of (14)–(18) decays exponentially as $t \to +\infty$.

Remark 2.1
C_b^3 denotes the set of bounded functions with bounded derivatives up to the third order. This condition is only made for the simplicity of the proof. However, the decay result still holds if (19) and (20) are satisfied only in a neighbourhood of the equilibrium state and the functions are taken in C^3. In this case a slight modification in the proof, as in References [24,25], is needed.

3. PROOF

Given a local solution, we multiply (14) by kdu_t, (15), by $kb\theta$ and (16) by gbq, integrate over I, using integration by parts, and add equations, to obtain

$$\frac{\mathrm{d}E_1(t)}{\mathrm{d}t} = -\int_0^1 bgq^2 \,\mathrm{d}x + R_1 \tag{26}$$

with

$$R_1 = \int_0^1 \frac{1}{2}(kd)_t u_t^2 - (akd)_x u_x u_t + \frac{1}{2}(akd)_t u_x^2 + (bkd)_x \theta u_t + \frac{1}{2}(kb)_t \theta^2$$
$$+ (kbg)_x q\theta + \frac{1}{2}(bg\tau)_t q^2 + (kd\alpha_1) qq_x u_t + (kd\alpha_2) qq_t \theta) \,\mathrm{d}x$$

By using (19) and (20), it is easy to see

$$|R_1| \leqslant \Gamma \alpha(t) \Lambda(t) \tag{27}$$

where Γ denotes a positive (possibly large) generic constant independent of u, θ, q, t. Then (26) becomes

$$\frac{\mathrm{d}E_1(t)}{\mathrm{d}t} \leqslant -\int_0^1 bgq^2 \,\mathrm{d}x + \Gamma \alpha(t) \Lambda(t) \tag{28}$$

Differentiating (14)–(16) with respect to t, we get

$$u_{ttt} - au_{xxt} - a_t u_{xx} + b\theta_{xt} + b_t \theta_x = \alpha_1 q_t q_x + \alpha_1 qq_{xt} + \alpha_{1t} qq_x \tag{29}$$

$$\theta_{tt} + gq_{xt} + g_t q_x + du_{ttx} + d_t u_{tx} = \alpha_2 q_t^2 + \alpha_{2t} q_t q + \alpha_2 q_{tt} q \tag{30}$$

$$\tau q_{tt} + \tau_t q_t + q_t + k\theta_{xt} + k_t \theta_x = 0 \tag{31}$$

In the same manner, multiplying (29) by kdu_{tt}, (30) by $kb\theta_t$, and (31) by gbq_t we obtain

$$\frac{\mathrm{d}E_2(t)}{\mathrm{d}t} = -\int_0^1 bgq_t^2 \,\mathrm{d}x + R_2 \tag{32}$$

with

$$\begin{aligned} R_2 = \int_0^1 (&kda_t u_{tt} u_{xx} - kdb_t \theta_x u_{tt} + \alpha_1 kdq_t q_x u_{tt} + \alpha_1 kdqq_{xt} u_{tt} - kbg_t q_x \theta_t \\ &- kbd_t u_{tx} \theta_t + \alpha_{1t} kdqq_x u_{tt} + \alpha_2 kbq_t^2 \theta_t + \alpha_{2t} kbq_t q\theta_t + \alpha_2 kbqq_{tt} \theta_t \\ &- bg\tau_t q_t^2 - k_t bg\theta_x q_t - (akd)_x u_{tt} u_{tx} + (bkd)_x \theta_t u_{tt} + (kbg)_x \theta_t q_t)\,\mathrm{d}x \\ &+ \frac{1}{2}\int_0^1 ((kd)_t u_{tt}^2 + (akd)_t u_{tx}^2 + (kb)_t \theta_t^2 + (bg\tau)_t q_t^2)\,\mathrm{d}x \end{aligned}$$

For additional estimates, we differentiate (29)–(31), with respect to t, to get

$$\begin{aligned} &u_{tttt} - au_{ttxx} - 2a_t u_{txx} + b\theta_{ttx} + 2b_t \theta_{tx} - a_{tt} u_{xx} + b_{tt} \theta_x \\ &\quad = 2\alpha_{1t} q_t q_x + \alpha_1 q_{tt} q_x + 2\alpha_1 q_t q_{tx} + 2\alpha_{1t} qq_{tx} + \alpha_1 qq_{ttx.+} \alpha_{1tt} qq_x \end{aligned} \tag{33}$$

$$\begin{aligned} &\theta_{ttt} + gq_{xtt} + 2g_t q_{tx} + du_{tttx} + 2d_t u_{ttx} + g_{tt} q_x + d_{tt} u_{tx} \\ &\quad = 2\alpha_{2t} q_t^2 + 3\alpha_2 q_t q_{tt} + \alpha_{2tt} q_t q + 2\alpha_{2t} q_{tt} q + \alpha_2 q_{ttt} q \end{aligned} \tag{34}$$

$$\tau q_{ttt} + 2\tau_t q_{tt} + \tau_{tt} q_t + q_{tt} + k\theta_{ttx} + 2k_t \theta_{tx} + k_{tt} \theta_x = 0 \tag{35}$$

We then multiply (33) by kdu_{ttt}, (34) by $kb\theta_{tt}$, and (35) by bgq_{tt} to have, by similar calculations

$$\frac{\mathrm{d}E_3(t)}{\mathrm{d}t} = -\int_0^1 bgq_{tt}^2 \,\mathrm{d}x + R_3 \tag{36}$$

where

$$\begin{aligned} R_3 = \int_0^1 &[(bkd)_x u_{ttt} \theta_{tt} + (kbg)_x \theta_{tt} q_{tt} - (adk)_x u_{ttx} u_{ttt}] + \int_0^1 kdu_{ttt} (2a_t u_{txx} \\ &- 2b_t \theta_{tx} + a_{tt} u_{xx} - b_{tt} \theta_x + 2\alpha_{1t} q_t q_x + \alpha_1 q_{tt} q_x + 2\alpha_1 q_t q_{tx} + 2\alpha_{1t} qq_{tx} \end{aligned}$$

$$+\alpha_1 q q_{ttx} + \alpha_{1tt} q q_x)\,\mathrm{d}x + \int_0^1 kb\theta_{tt}(-2g_t q_{tx} - 2d_t u_{ttx} - g_{tt} q_x - d_{tt} u_{tx} + 2\alpha_{2t} q_t^2$$

$$+3\alpha_2 q_t q_{tt} + \alpha_{2tt} q_t q + 2\alpha_{2t} q_{tt} q + \alpha_2 q_{ttt} q)\,\mathrm{d}x - \int_0^1 bgq_{tt}(2\tau_t q_{tt} + \tau_{tt} q_t$$

$$+2k_t\theta_{tx} + k_{tt}\theta_x)\,\mathrm{d}x + \frac{1}{2}\int_0^1 [(kd)_t u_{ttt}^2 + (akd)_t u_{ttx} + (kb)_t \theta_{tt}^2 + (bg\tau)_t q_{tt}^2]\,\mathrm{d}x \tag{37}$$

By the same manner as in (28) and (32) we arrive at

$$\frac{\mathrm{d}E_3(t)}{\mathrm{d}t} \leqslant -\int_0^1 bgq_{tt}^2\,\mathrm{d}x + \Gamma(\alpha(t) + \alpha^2(t) + \alpha^3(t))\Lambda(t) \tag{38}$$

We then use (16) and $(a+b)^2 \leqslant 2(a^2+b^2)$, to get

$$\int_0^1 \theta_x^2\,\mathrm{d}x \leqslant \int_0^1 \frac{2\tau^2}{k^2} q_t^2\,\mathrm{d}x + \int_0^1 \frac{2}{k^2} q^2\,\mathrm{d}x \tag{39}$$

A differentiation of (16) with respect to t then leads to

$$\tau q_{tt} + q_t + k\theta_{xt} + \tau_t q_t + k_t \theta_x = 0 \tag{40}$$

Multiply (40) by θ_{xt} and integrate over I to find

$$\int_0^1 k\theta_{xt}^2\,\mathrm{d}x = -\int_0^1 (\tau q_{tt} + q_t + \tau_t q_t + k_t\theta_x)\theta_{xt}\,\mathrm{d}x \tag{41}$$

and thanks to Young's inequality and (19), we obtain

$$\int_0^1 k\theta_{xt}^2\,\mathrm{d}x \leqslant C\int_0^1 (q_{tt}^2 + q_t^2)\,\mathrm{d}x + \widetilde{R_1} \tag{42}$$

where

$$\widetilde{R_1} = \int_0^1 (\tau_t q_t \theta_{xt} + k_t\theta_x\theta_{xt})\,\mathrm{d}x$$

Then from (39) and (42) we can write

$$\int_0^1 (\theta_x^2 + \theta_{xt}^2)\,\mathrm{d}x \leqslant C\int_0^1 (q_{tt}^2 + q_t^2 + q^2)\,\mathrm{d}x + \widetilde{R_1} \tag{43}$$

Multiplying Equation (14) by $(1/a)u_{xx}$ integrating over I and using integration by parts, we have

$$\int_0^1 u_{xx}^2\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{a} u_{tx}u_x\,\mathrm{d}x + \int_0^1 \left(\frac{1}{a}\right)_t u_{tx}u_x\,\mathrm{d}x + \int_0^1 \frac{1}{a}u_{tx}^2\,\mathrm{d}x$$

$$-\int_0^1 \left(\frac{1}{a}\right)_x u_{tt}u_x\,\mathrm{d}x + \int_0^1 \frac{b}{a}\theta_x u_{xx}\,\mathrm{d}x - \int_0^1 \frac{\alpha_1}{a} qq_x u_{xx}\,\mathrm{d}x$$

Young's inequality then gives

$$\int_0^1 u_{xx}^2\,\mathrm{d}x \leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{a}\,u_{tx}u_x\,\mathrm{d}x + \int_0^1 \frac{1}{a}\,u_{tx}^2\,\mathrm{d}x + \int_0^1 \frac{3b^2}{4a^2}\,\theta_x^2\,\mathrm{d}x + \frac{1}{3}\int_0^1 u_{xx}^2\,\mathrm{d}x$$
$$+ \int_0^1 \left(\frac{1}{a}\right)_t u_{tx}u_x\,\mathrm{d}x - \int_0^1 \left(\frac{1}{a}\right)_x u_{tt}u_x\,\mathrm{d}x - \int_0^1 \frac{\alpha_1}{a}\,qq_xu_{xx}\,\mathrm{d}x$$

which implies

$$\frac{2}{3}\int_0^1 u_{xx}^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{a}u_{tx}u_{xx}\,\mathrm{d}x \leqslant \int_0^1 \frac{1}{a}\,u_{tx}^2\,\mathrm{d}x$$
$$+ \int_0^1 \frac{3b^2}{4a^2}\,\theta_x^2\,\mathrm{d}x + \int_0^1 \left(\frac{1}{a}\right)_t u_{tx}u_x\,\mathrm{d}x - \int_0^1 \left(\frac{1}{a}\right)_x u_{tt}u_x\,\mathrm{d}x - \int_0^1 \frac{\alpha_1}{a}qq_xu_{xx}\,\mathrm{d}x \qquad (44)$$

Multiplying (15) by $(3/ad)u_{tx}$ integrating over I and using integration by parts, we get

$$\int_0^1 \frac{3}{a}\,u_{tx}^2\,\mathrm{d}x = \int_0^1 \left(\frac{3}{ad}\right)_x \theta_tu_t\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{3}{ad}\,\theta_xu_t\,\mathrm{d}x - \int_0^1 \left(\frac{3}{ad}\right)_t \theta_xu_t\,\mathrm{d}x$$
$$- \int_0^1 \frac{3}{d}\,\theta_xu_{xx}\,\mathrm{d}x + \int_0^1 \frac{3b}{ad}\,\theta_x^2\,\mathrm{d}x - \int_0^1 \frac{3\alpha_1}{ad}\,\theta_xqq_x - \left[\frac{3g}{ad}\,qu_{tx}\right]_{x=0}^{x=1}$$
$$+ \int_0^1 \left(\frac{3g}{ad}\right)_x u_{tx}q\,\mathrm{d}x - \int_0^1 \left(\frac{3g}{ad}\right)_t u_{xx}q\,\mathrm{d}x - \int_0^1 \frac{3g}{ad}\,u_{xx}q_t\,\mathrm{d}x$$
$$+ \int_0^1 \frac{3\alpha_2}{ad}\,qq_tu_{tx}\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left\{\frac{3g}{a^2d}\,qu_{tt} + \frac{3gb}{a^2d}\,q\theta_x - \frac{3g\alpha_1}{a^2d}\,q^2q_x\right\}\mathrm{d}x$$

Then we obtain, with the help of (16) and Young's inequality,

$$\int_0^1 \frac{3}{a}\,u_{tx}^2\,\mathrm{d}x \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left\{\frac{3}{ad}\,\theta_xu_t + \frac{3g}{a^2d}\,qu_{tt}\,\mathrm{d}x - \frac{3gb\tau}{a^2dk}\,qq_t\,\mathrm{d}x - \frac{3g\alpha_1}{a^2d}\,q^2q_x - \frac{3gb}{a^2dk}q^2\right\}\mathrm{d}x$$
$$+ \int_0^1 \left(\frac{3}{ad}\right)_x \theta_tu_t\,\mathrm{d}x - \int_0^1 \left(\frac{3}{ad}\right)_t \theta_xu_t\,\mathrm{d}x + \int_0^1 \frac{3b}{ad}\,\theta_x^2\,\mathrm{d}x - \int_0^1 \frac{3\alpha_1}{ad}\,\theta_xqq_x$$
$$- \left[\frac{3g}{ad}\,qu_{tx}\right]_{x=0}^{x=1} + \int_0^1 \left(\frac{3g}{ad}\right)_x u_{tx}q\,\mathrm{d}x - \int_0^1 \left(\frac{3g}{ad}\right)_t u_{xx}q\,\mathrm{d}x + \frac{1}{12}\int_0^1 u_{xx}^2\,\mathrm{d}x$$
$$+ \int_0^1 \frac{27g^2}{a^2d^2}\,q_t^2\,\mathrm{d}x + \frac{1}{12}\int_0^1 u_{xx}^2\,\mathrm{d}x + \int_0^1 \frac{27}{d^2}\,\theta_x^2\,\mathrm{d}x + \int_0^1 \frac{3\alpha_2}{ad}\,qq_tu_{tx}\,\mathrm{d}x \qquad (45)$$

A combination of (44) and (45) then yields

$$\int_0^1 \frac{2}{a} u_{tx}^2 \,\mathrm{d}x + \int_0^1 \frac{1}{2} u_{xx}^2 \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left(\frac{1}{a} u_{tx} u_x - \frac{3g}{a^2 d} q u_{tt} \,\mathrm{d}x \right.$$
$$\left. + \frac{3gb\tau}{a^2 dk} q q_t + \frac{3gb}{a^2 dk} q^2 + \frac{3}{akd} q u_t + \frac{3\tau}{akd} q_t u_t \right) \mathrm{d}x$$
$$\leqslant \int_0^1 \left(\frac{3b^2}{4a^2} + \frac{3b}{ad} + \frac{27}{d^2} \right) \theta_x^2 \,\mathrm{d}x + \int_0^1 \frac{27g^2}{a^2 d^2} q_t^2 \,\mathrm{d}x - \left[\frac{3g}{ad} q u_{tx} \right]_{x=0}^{x=1} + R_4 \tag{46}$$

where

$$R_4 = \int_0^1 \left(\frac{1}{a} \right)_t u_{tx} u_x \,\mathrm{d}x - \int_0^1 \left(\frac{1}{a} \right)_x u_{tt} u_x \,\mathrm{d}x - \int_0^1 \frac{\alpha_1}{a} q q_x u_{xx} \,\mathrm{d}x$$
$$+ \int_0^1 \left(\frac{3}{ad} \right)_x \theta_t u_t \,\mathrm{d}x - \int_0^1 \left(\frac{3}{ad} \right)_t \theta_x u_t \,\mathrm{d}x - \int_0^1 \frac{3\alpha_1}{ad} \theta_x q q_x$$
$$+ \int_0^1 \left(\frac{3g}{ad} \right)_x u_{tx} q \,\mathrm{d}x - \int_0^1 \left(\frac{3g}{ad} \right)_t u_{xx} q \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \frac{3g\alpha_1}{a^2 d} q^2 q_x \,\mathrm{d}x$$
$$+ \int_0^1 \frac{3\alpha_2}{ad} q q_t u_{tx} \,\mathrm{d}x \tag{47}$$

Similarly, multiplying Equation (29) by $(1/a)u_{xxt}$ we get, after integration over I and use of Young's inequality,

$$\int_0^1 u_{xxt}^2 \,\mathrm{d}x \leqslant -\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \frac{1}{a} u_{ttx} u_{xt} \,\mathrm{d}x + \int_0^1 \frac{1}{a} u_{ttx}^2 \,\mathrm{d}x + \int_0^1 \frac{3b^2}{4a^2} \theta_{xt}^2 \,\mathrm{d}x + \frac{1}{3} \int_0^1 u_{xxt}^2 \,\mathrm{d}x$$
$$+ \int_0^1 \left(\frac{1}{a} \right)_t u_{ttx} u_{xt} \,\mathrm{d}x - \int_0^1 \left(\frac{1}{a} \right)_x u_{ttt} u_{xt} \,\mathrm{d}x - \int_0^1 \frac{a_t}{a} u_{xx} u_{xxt} \,\mathrm{d}x$$
$$+ \int_0^1 \frac{b_t}{a} \theta_x u_{xxt} \,\mathrm{d}x - \int_0^1 \frac{\alpha_1}{a} q_t q_x u_{xxt} \,\mathrm{d}x - \int_0^1 \frac{\alpha_1}{a} q q_{xt} u_{xxt} \,\mathrm{d}x$$
$$- \int_0^1 \frac{\alpha_{1t}}{a} q q_x u_{xxt} \,\mathrm{d}x \tag{48}$$

Also multiply (30) by $(3/ad)u_{ttx}$ to get, by the same manner,

$$\int_0^1 \frac{3}{a} u_{ttx}^2 \,\mathrm{d}x = -\int_0^1 \frac{3}{ad} \theta_{tt} u_{ttx} \,\mathrm{d}x - \int_0^1 \frac{3g}{ad} u_{ttx} q_{xt} \,\mathrm{d}x - \int_0^1 \frac{3g_t}{ad} q_x u_{ttx} \,\mathrm{d}x$$

$$
\begin{aligned}
&-\int_0^1 \frac{3d_t}{ad}\, u_{tx}u_{ttx}\, \mathrm{d}x + \int_0^1 \frac{3\alpha_2}{ad}\, q_t^2 u_{ttx}\, \mathrm{d}x + \int_0^1 \frac{3\alpha_{2t}}{ad}\, q_t q u_{ttx}\, \mathrm{d}x \\
&+\int_0^1 \frac{3\alpha_2}{ad}\, q_{tt} q u_{ttx}\, \mathrm{d}x
\end{aligned} \tag{49}
$$

By multiplying (29) by $(3/ad)\theta_{xt}$ and integrating over I, we have

$$
\begin{aligned}
-\int_0^1 \frac{3}{ad}\,\theta_{tt}u_{ttx}\,\mathrm{d}x = &\int_0^1 \left(\frac{3}{ad}\right)_x \theta_{tt}u_{tt}\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{3}{ad}\,\theta_{tx}u_{tt}\,\mathrm{d}x - \int_0^1 \left(\frac{3}{ad}\right)_t \theta_{tx}u_{tt}\,\mathrm{d}x \\
&-\int_0^1 \frac{3}{d}\,\theta_{tx}u_{xxt}\,\mathrm{d}x - \int_0^1 \frac{3a_t}{ad}\,\theta_{tx}u_{xx}\,\mathrm{d}x + \int_0^1 \frac{3b}{ad}\,\theta_{tx}^2\,\mathrm{d}x \\
&+\int_0^1 \frac{3b_t}{ad}\,\theta_{tx}\theta_x\,\mathrm{d}x - \int_0^1 \frac{3\alpha_1}{ad}\,\theta_{tx}q_t q_x\,\mathrm{d}x - \int_0^1 \frac{3\alpha_1}{ad}\,\theta_{tx}q q_{xt}\,\mathrm{d}x \\
&-\int_0^1 \frac{3\alpha_{1t}}{ad}\,\theta_{tx}q q_x\,\mathrm{d}x
\end{aligned} \tag{50}
$$

Using integration by parts, we easily see

$$
\begin{aligned}
-\int_0^1 \frac{3g}{ad}\,u_{ttx}q_{xt}\,\mathrm{d}x = &-\left[\frac{3g}{ad}q_t u_{ttx}\right]_{x=0}^{x=1} + \int_0^1 \left(\frac{3g}{ad}\right)_x u_{ttx}q_t\,\mathrm{d}x \\
&+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{3g}{ad}\,u_{txx}q_t\,\mathrm{d}x - \int_0^1 \left(\frac{3g}{ad}\right)_t u_{txx}q_t\,\mathrm{d}x - \int_0^1 \frac{3g}{ad}\,u_{txx}q_{tt}\,\mathrm{d}x
\end{aligned}
$$

Again multiply (29) by $(3g/a^2d)q_t$ and integrate over I to obtain

$$
\begin{aligned}
-\int_0^1 \frac{3g}{ad}\,u_{ttx}q_{xt}\,\mathrm{d}x = &-\left[\frac{3g}{ad}q_t u_{ttx}\right]_{x=0}^{x=1} + \int_0^1 \left(\frac{3g}{ad}\right)_x u_{ttx}q_t\,\mathrm{d}x \\
&-\int_0^1 \left(\frac{3g}{ad}\right)_t u_{txx}q_t\,\mathrm{d}x - \int_0^1 \frac{3g}{ad}\,u_{txx}q_{tt}\,\mathrm{d}x \\
&+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\frac{3g}{a^2d}\,q_t u_{ttt} - \frac{3gb}{a^2d}q_t\theta_{xt} - \frac{3\alpha_1 g}{a^2d}\,q_t q q_{xt}\right. \\
&\left.-\frac{3ga_t}{a^2d}\,q_t u_{xx} + \frac{3gb_t}{a^2d}\,q_t\theta_x - \frac{3\alpha_1 g}{a^2d}\,q_t^2 q_x - \frac{3\alpha_{1t} g}{a^2d}\,q_t q q_x\right)\mathrm{d}x
\end{aligned} \tag{51}
$$

Combining (49)–(51) we arrive at

$$
\begin{aligned}
\int_0^1 \frac{3}{a} u_{ttx}^2 \,\mathrm{d}x = & \int_0^1 \left(\frac{3}{ad}\right)_x \theta_{tt} u_{tt} \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \frac{3}{ad} \theta_{tx} u_{tt} \,\mathrm{d}x - \int_0^1 \left(\frac{3}{ad}\right)_t \theta_{tx} u_{tt} \,\mathrm{d}x \\
& - \int_0^1 \frac{3}{d} \theta_{tx} u_{xxt} \,\mathrm{d}x - \int_0^1 \frac{3a_t}{ad} \theta_{tx} u_{xx} \,\mathrm{d}x + \int_0^1 \frac{3b}{ad} \theta_{tx}^2 \,\mathrm{d}x \\
& + \int_0^1 \frac{3b_t}{ad} \theta_{tx} \theta_x \,\mathrm{d}x - \int_0^1 \frac{3\alpha_1}{ad} \theta_{tx} q_t q_x \,\mathrm{d}x - \int_0^1 \frac{3\alpha_1}{ad} \theta_{tx} q q_{xt} \,\mathrm{d}x \\
& - \int_0^1 \frac{3\alpha_{1t}}{ad} \theta_{tx} q q_x \mathrm{d} - \left[\frac{3g}{ad} q_t u_{ttx}\right]_{x=0}^{x=1} + \int_0^1 \left(\frac{3g}{ad}\right)_x u_{ttx} q_t \,\mathrm{d}x \\
& - \int_0^1 \left(\frac{3g}{ad}\right)_t u_{txx} q_t \,\mathrm{d}x - \int_0^1 \frac{3g}{ad} u_{txx} q_{tt} \,\mathrm{d}x - \int_0^1 \frac{3g_t}{ad} q_x u_{ttx} \,\mathrm{d}x \\
& - \int_0^1 \frac{3d_t}{ad} u_{tx} u_{ttx} \,\mathrm{d}x + \int_0^1 \frac{3\alpha_2}{ad} q_t^2 u_{ttx} \,\mathrm{d}x + \int_0^1 \frac{3\alpha_{2t}}{ad} q_t q u_{ttx} \,\mathrm{d}x \\
& + \int_0^1 \frac{3\alpha_2}{ad} q_{tt} q u_{ttx} \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left(\frac{3g}{a^2 d} q_t u_{ttt} - \frac{3gb}{a^2 d} q_t \theta_{xt} - \frac{3\alpha_1 g}{a^2 d} q_t q q_{xt} \right. \\
& \left. - \frac{3g a_t}{a^2 d} q_t u_{xx} + \frac{3g b_t}{a^2 d} q_t \theta_x - \frac{3\alpha_1 g}{a^2 d} q_t^2 q_x - \frac{3\alpha_{1t} g}{a^2 d} q_t q q_x \right) \mathrm{d}x
\end{aligned} \tag{52}
$$

Using Equation (31) we can write

$$
\begin{aligned}
\int_0^1 \frac{3gb}{a^2 d} q_t \theta_{xt} \,\mathrm{d}x = & - \int_0^1 \frac{3gb\tau}{a^2 dk} q_t q_{tt} \,\mathrm{d}x - \int_0^1 \frac{3gb\tau_t}{a^2 dk} q_t^2 \,\mathrm{d}x \\
& - \int_0^1 \frac{3gb}{a^2 dk} q_t^2 \,\mathrm{d}x - \int_0^1 \frac{3gbk_t}{a^2 dk} q_t \theta_x \,\mathrm{d}x
\end{aligned} \tag{53}
$$

Young's inequality gives

$$
- \int_0^1 \frac{3}{d} \theta_{tx} u_{xxt} \,\mathrm{d}x \leqslant \frac{1}{12} \int_0^1 u_{txx}^2 \,\mathrm{d}x + \int_0^1 \frac{27}{d^2} \theta_{tx}^2 \,\mathrm{d}x \tag{54}
$$

and

$$
- \int_0^1 \frac{3g}{ad} u_{xxt} q_{tt} \,\mathrm{d}x \leqslant \frac{1}{12} \int_0^1 u_{txx}^2 \,\mathrm{d}x + \int_0^1 \frac{27g^2}{a^2 d^2} q_{tt}^2 \,\mathrm{d}x \tag{55}
$$

Taking into account (53)–(55), estimate (52) becomes

$$\int_0^1 \frac{3}{a}u_{ttx}^2\,\mathrm{d}x = \int_0^1 \left(\frac{1}{6}u_{txx}^2 + \frac{27g^2}{a^2d^2}q_{tt}^2\,\mathrm{d}x + \left(\frac{27}{d^2}+\frac{3b}{ad}\right)\theta_{tx}^2 - \left(\frac{3}{ad}\right)_t\theta_{tx}u_{tt} - \frac{3a_t}{ad}\theta_{tx}u_{xx}\right.$$
$$+\frac{3b_t}{ad}\theta_{tx}\theta_x - \frac{3\alpha_1}{ad}\theta_{tx}q_tq_x - \frac{3\alpha_1}{ad}\theta_{tx}qq_{xt} - \frac{3\alpha_{1t}}{ad}\theta_{tx}qq_x + \left(\frac{3g}{ad}\right)_x u_{ttx}q_t$$
$$-\left(\frac{3g}{ad}\right)_t u_{txx}q_t - \frac{3g_t}{ad}q_xu_{ttx}\,\mathrm{d}x - \frac{3d_t}{ad}u_{tx}u_{ttx} + \frac{3\alpha_2}{ad}q_t^2u_{ttx}$$
$$\left.+\frac{3\alpha_{2t}}{ad}q_tqu_{ttx} + \frac{3\alpha_2}{ad}q_{tt}qu_{ttx}\right)\mathrm{d}x - \left[\frac{3g}{ad}q_tu_{ttx}\right]_{x=0}^{x=1}$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(-\frac{3\tau}{adk}u_{tt}q_{tt} - \frac{3\tau_t}{adk}u_{tt}q_t - \frac{3}{adk}u_{tt}q_t - \frac{3k_t}{adk}u_{tt}\theta_x + \frac{3g}{a^2d}q_tu_{ttt}\right.$$
$$-\frac{3gb\tau}{a^2dk}q_tq_{tt} - \frac{3gb\tau_t}{a^2dk}q_t^2 - \frac{3gb}{a^2dk}q_t^2 - \frac{3gbk_t}{a^2dk}q_t\theta_x - \frac{3\alpha_1 g}{a^2d}q_tqq_{xt}$$
$$\left.-\frac{3ga_t}{a^2d}q_tu_{xx} + \frac{3gb_t}{a^2d}q_t\theta_x - \frac{3\alpha_1 g}{a^2d}q_t^2q_x - \frac{3\alpha_{2t}g}{a^2d}q_tqq_x\right)\mathrm{d}x \tag{56}$$

Combining (48) and (56) we obtain

$$\int_0^1 \frac{2}{a}u_{ttx}^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 u_{xxt}^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\frac{1}{a}u_{ttx}u_{xt}\,\mathrm{d}x - \frac{3g}{a^2d}q_tu_{ttt}\right.$$
$$+\frac{3gb\tau}{a^2dk}q_tq_{tt} + \frac{3bg}{a^2dk}q_t^2 + \frac{3\tau}{adk}u_{tt}q_{tt} + \frac{3}{adk}u_{tt}q_t$$
$$\left.-\frac{3\alpha_1}{a^2d}q_tq\theta_{tt} - \frac{3\alpha_1}{a^2}q_tqu_{ttx} - \frac{3\alpha_1\alpha_2}{a^2d}q_tq^2q_{tt}\right)\mathrm{d}x$$
$$\leqslant -\left[\frac{3g}{ad}q_tu_{ttx}\right]_{x=0}^{x=1} + \int_0^1\left(\frac{3b^2}{4a^2}+\frac{3b}{ad}+\frac{27}{d^2}\right)\theta_{xt}^2\,\mathrm{d}x$$
$$+\int_0^1 \frac{27g^2}{a^2d^2}q_{tt}^2\,\mathrm{d}x + R_5 \tag{57}$$

where

$$R_5 = \int_0^1\left(\frac{1}{a}\right)_t u_{ttx}u_{xt}\,\mathrm{d}x - \int_0^1\left(\frac{1}{a}\right)_x u_{ttt}u_{xt}\,\mathrm{d}x - \int_0^1\frac{a_t}{a}u_{xx}u_{xxt}\,\mathrm{d}x + \int_0^1\frac{b_t}{a}\theta_xu_{xxt}\,\mathrm{d}x$$
$$-\int_0^1\frac{\alpha_1}{a}q_tq_xu_{xxt}\,\mathrm{d}x - \int_0^1\frac{\alpha_1}{a}qq_{xt}u_{xxt}\,\mathrm{d}x - \int_0^1\frac{\alpha_{1t}}{a}qq_xu_{xxt}\,\mathrm{d}x$$

$$
\begin{aligned}
&-\int_0^1 \left(\frac{3}{ad}\right)_t \theta_{tx}u_{tt}\,\mathrm{d}x - \int_0^1 \frac{3a_t}{ad}\,\theta_{tx}u_{xx}\,\mathrm{d}x + \int_0^1 \frac{3b_t}{ad}\,\theta_{tx}\theta_x\,\mathrm{d}x - \int_0^1 \frac{3\alpha_1}{ad}\,\theta_{tx}q_t q_x\,\mathrm{d}x \\
&-\int_0^1 \frac{3\alpha_1}{ad}\,\theta_{tx}qq_{xt}\,\mathrm{d}x - \int_0^1 \frac{3\alpha_{1t}}{ad}\,\theta_{tx}qq_x\,\mathrm{d}x + \int_0^1 \left(\frac{3g}{ad}\right)_x u_{ttx}q_t\,\mathrm{d}x \\
&-\int_0^1 \left(\frac{3g}{ad}\right)_t u_{txx}q_t\,\mathrm{d}x - \int_0^1 \frac{3g_t}{ad}\,q_x u_{ttx}\,\mathrm{d}x - \int_0^1 \frac{3d_t}{ad}\,u_{tx}u_{ttx}\,\mathrm{d}x \\
&+\int_0^1 \frac{3\alpha_2}{ad}\,q_t^2 u_{ttx}\,\mathrm{d}x + \int_0^1 \frac{3\alpha_{2t}}{ad}\,q_t q u_{ttx}\,\mathrm{d}x + \int_0^1 \frac{3\alpha_2}{ad}\,q_{tt}qu_{ttx}\,\mathrm{d}x \\
&+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(-\frac{3\tau_t}{adk}\,u_{tt}q_t - \frac{3k_t}{adk}\,u_{tt}\theta_x - \frac{3bg\tau_t}{a^2dk}\,q_t^2 - \frac{3bgk_t}{a^2dk}\,q_t\theta_x\right. \\
&-\frac{3ga_t}{a^2d}\,q_t u_{xx} + \frac{3gb_t}{a^2d}\,q_t\theta_x - \frac{3\alpha_1 g}{a^2d}\,q_t^2 q_x - \frac{3\alpha_{2t}g}{a^2d}\,q_t qq_x \\
&\left.+\frac{3\alpha_1}{a^2d}\,q_t q\left(-g_t q_x - d_t u_{tx} + \alpha_2 q_t^2 + \alpha_{2t}q_t q - \frac{3\alpha_{2t}g}{a^2d}\,q_t qq_x\right)\right)\mathrm{d}x
\end{aligned}
$$

Now we multiply (14) by u_{tt}, integrate over I and use Young's inequality to arrive at

$$
\int_0^1 u_{tt}^2\,\mathrm{d}x \leqslant 2\int_0^1 a^2u_{xx}^2\,\mathrm{d}x + 2\int_0^1 b^2\theta_x^2\,\mathrm{d}x + 2\int_0^1 \alpha_1 qq_x u_{tt}\,\mathrm{d}x
$$

and use (16) together with Poincare's inequality for u_t and θ to obtain

$$
\begin{aligned}
\int_0^1 (u_{tt}^2 + u_t^2 + \theta^2)\,\mathrm{d}x \leqslant 2\int_0^1 a^2u_{xx}^2\,\mathrm{d}x + \int_0^1 (2b^2+1)\left(\frac{2\tau^2}{k^2}\,q_t^2 + \frac{2}{k^2}q^2\right)\mathrm{d}x \\
+2\int_0^1 \alpha_1 qq_x u_{tt}\,\mathrm{d}x + \int_0^1 u_{tx}^2\,\mathrm{d}x \qquad (58)
\end{aligned}
$$

Similar treatment to (29) and (31) yields

$$
\begin{aligned}
\int_0^1 (u_{ttt}^2 + u_{tt}^2 + \theta_t^2)\,\mathrm{d}x \leqslant 2\int_0^1 a^2u_{xxt}^2\,\mathrm{d}x + \int_0^1 (2b^2+1)\left(\frac{2\tau^2}{k^2}\,q_{tt}^2 + \frac{2}{k^2}\,q_t^2\right)\mathrm{d}x \\
+\int_0^1 u_{ttx}^2\,\mathrm{d}x + 2\int_0^1 (a_t u_{xx}u_{ttt} - b_t\theta_x u_{ttt} + \alpha_1 q_t q_x u_{ttt} \\
+\alpha_1 qq_{xt}u_{ttt} + \alpha_{1t}qq_x u_{ttt} - \frac{\tau_t}{k}\,q_t\theta_{tx} - \frac{k_t}{k}\,\theta_x\theta_{tx})\,\mathrm{d}x \qquad (59)
\end{aligned}
$$

Therefore (58) together with (59) give

$$\int_0^1 (u_{ttt}^2 + u_{tt}^2 + u_t^2 + \theta^2 + \theta_t^2)\,\mathrm{d}x \leqslant C\int_0^1 (u_{xx}^2 + u_{xxt}^2 + u_{tx}^2 + u_{ttx}^2 + q_{tt}^2 + q_t^2 + q^2)\,\mathrm{d}x + R_6 \tag{60}$$

where C is a constant and

$$R_6 = 2\int_0^1 \Big(\alpha_1 q q_x u_{tt} + a_t u_{xx} u_{ttt} - b_t \theta_x u_{ttt} + \alpha_1 q_t q_x u_{ttt} + \alpha_1 q q_{xt} u_{ttt} + \alpha_{1t} q q_x u_{ttt} - \frac{\tau_t}{k} q_t \theta_{tx} - \frac{k_t}{k}\theta_x \theta_{tx}\Big)\,\mathrm{d}x$$

Multiplying (14) by u using Poincaré's inequality and Young's inequality we easily arrive at

$$\int_0^1 u_x^2\,\mathrm{d}x \leqslant C\int_0^1 (u_{tt}^2 + \theta_x^2)\,\mathrm{d}x + R_7 \tag{61}$$

where

$$R_7 = \int_0^1 (-a_x u u_{xx} + \alpha_1 q q_x u)\,\mathrm{d}x$$

Also multiplication of (14) by θ_t integration over I, and use of Young's inequality leads to

$$\int_0^1 \theta_t^2\,\mathrm{d}x \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 g q \theta_x\,\mathrm{d}x + \int_0^1 \Big(\frac{g^2}{2} q_t^2 + \frac{1}{2}\theta_x^2 + \frac{d^2}{2} u_{tx}^2 + \frac{1}{2}\theta_t^2 - g_t q \theta_x + g_x \theta_t q + \alpha_2 q q_t \theta_t\Big)\,\mathrm{d}x \tag{62}$$

A similar treatment to (30) gives

$$\int_0^1 \theta_{tt}^2\,\mathrm{d}x \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 g\theta_{tx} q_t\,\mathrm{d}x + \int_0^1 \Big(-g_t \theta_{tx} q_t + g_x \theta_{tt} q_t + \frac{d^2}{2} u_{ttx}^2 + \frac{1}{2}\theta_{tt}^2 + \frac{g^2}{2} q_{tt}^2 + \frac{1}{2}\theta_{tx}^2 - g_t \theta_{tt} q_x - d_t \theta_{tt} u_{tx} + \alpha_2 q_t^2 \theta_{tt} + \alpha_{2t} q q_t \theta_{tt}\Big)\,\mathrm{d}x \tag{63}$$

From (62), (63), (19), and (20) we obtain

$$\int_0^1 (\theta_{tt}^2 + \theta_t^2)\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (2gq\theta_x + 2gq_t\theta_{xt})\,\mathrm{d}x$$
$$\leqslant C\int_0^1 (q_t^2 + \theta_x^2 + q_{tt}^2 + \theta_{tx}^2)\,\mathrm{d}x + \int_0^1 d^2(u_{ttx}^2 + u_{tx}^2)\,\mathrm{d}x + R_8 \tag{64}$$

where

$$R_8 = 2\int_0^1 (-g_t q\theta_x + g_x\theta_t q + \alpha_2 qq_t\theta_t - d_t\theta_{tt}u_{tx} + \alpha_2 q_t^2\theta_{tt}$$
$$- g_t\theta_{tx}q_t + g_x\theta_{tt}q_t - g_t\theta_{tt}q_x + \alpha_{2t}qq_t\theta_{tt})\,\mathrm{d}x \tag{65}$$

The boundary terms in (46) and (56) are treated as in Reference [5]. In fact, by Young's inequality we have

$$\left[\frac{3g}{ad}\,qu_{tx}\right]_{x=0}^{x=1} \leqslant \left[\frac{9g^2}{4a^3d^2\varepsilon}\,q^2 + \varepsilon au_{tx}^2\right]_{x=0}^{x=1} \leqslant \left|\left[\frac{9g^2}{4a^3d^2\varepsilon}\,q^2\right]_{x=0}^{x=1}\right| + |[\varepsilon au_{tx}^2]_{x=0}^{x=1}|$$
$$\leqslant \frac{C_t}{\varepsilon}(q^2(1) + q^2(0)) + |[\varepsilon au_{tx}^2]_{x=0}^{x=1}|$$

where

$$C_t = \mathrm{Max}\left\{\left[\frac{9g^2}{4a^3d^2}\right]_{x=0},\quad \left[\frac{9g^2}{4a^3d^2}\right]_{x=1}\right\}$$

By the imbedding of $W^{1,1}$ in L^∞ we have

$$|q(x)|^2 \leqslant \int_0^1 (q^2 + (q^2)_x)\,\mathrm{d}x, \quad 0 \leqslant x \leqslant 1$$

We exploit Young's inequality to get

$$|q(x)|^2 \leqslant 2\left(1 + \frac{1}{\varepsilon^2}\right)\int_0^1 q^2\,\mathrm{d}x + 2\varepsilon^2\int_0^1 q_x^2\,\mathrm{d}x, \quad 0 \leqslant x \leqslant 1$$

Therefore,

$$\frac{C_t}{\varepsilon}(q^2(1) + q^2(0)) \leqslant \frac{4C_t}{\varepsilon}\left(1 + \frac{1}{\varepsilon^2}\right)\int_0^1 q^2\,\mathrm{d}x + 4C_t\varepsilon\int_0^1 q_x^2\,\mathrm{d}x$$

Using (15) to obtain

$$\frac{C_t}{\varepsilon}(q^2(1)+q^2(0)) \leqslant \frac{4C_t}{\varepsilon}\left(1+\frac{1}{\varepsilon^2}\right)\int_0^1 q^2\,\mathrm{d}x + 4C_t\varepsilon\int_0^1\left(-\frac{1}{g}\theta_t q_x - \frac{d}{g}u_{tx}q_x + \frac{\alpha_2}{g}\,qq_tq_x\right)\mathrm{d}x$$

Again Young's inequality allows us to write

$$\left[\frac{3g}{ad}\,qu_{tx}\right]_{x=0}^{x=1} \leqslant \frac{4C_t}{\varepsilon^3}(1+\varepsilon^2)\int_0^1 q^2\,\mathrm{d}x + \int_0^1 \frac{36C_t\varepsilon}{a^2d^2}\,\theta_t^2\,\mathrm{d}x + \int_0^1 \frac{36C_t\varepsilon}{a^2}\,u_{tx}^2\,\mathrm{d}x + \int_0^1 \frac{2C_t\varepsilon a^2d^2}{9}\,q_x^2\,\mathrm{d}x + \int_0^1 \frac{4C_t\varepsilon\alpha_2}{g}\,qq_tq_x + |[\varepsilon a u_{tx}^2]_{x=0}| + |[\varepsilon a u_{tx}^2]_{x=1}| \tag{66}$$

By the same manner we can estimate the boundary term in (56) as follows:

$$\left[\frac{3g}{ad}q_tu_{ttx}\right]_{x=0}^{x=1} \leqslant \left[\frac{9g^2}{4a^3d^2\varepsilon}q_t^2 + \varepsilon a u_{ttx}^2\right]_{x=0}^{x=1} \leqslant \frac{C_t}{\varepsilon}(q_t^2(1)+q_t^2(0)) + |[\varepsilon a u_{ttx}^2]_{x=0}^{x=1}|$$

then we obtain

$$\frac{C_t}{\varepsilon}\,(q_t^2(1)+q_t^2(0)) \leqslant \frac{4C_t}{\varepsilon}\left(1+\frac{1}{\varepsilon^2}\right)\int_0^1 q_t^2\,\mathrm{d}x + 4C_t\varepsilon\int_0^1 q_{tx}^2\,\mathrm{d}x$$

Using (30) and Young's inequality, we get

$$\left[\frac{3g}{ad}\,q_tu_{ttx}\right]_{x=0}^{x=1} \leqslant \frac{4C_t}{\varepsilon^3}\,(1+\varepsilon^2)\int_0^1 q_t^2\,\mathrm{d}x + \int_0^1 \frac{36C_t\varepsilon}{a^2d^2}\,\theta_{tt}^2\,\mathrm{d}x + \int_0^1 \frac{36C_t\varepsilon}{a^2}\,u_{ttx}^2\,\mathrm{d}x + \int_0^1 \frac{2C_t\varepsilon a^2d^2}{9}\,q_{tx}^2\,\mathrm{d}x + |[\varepsilon a u_{ttx}^2]_{x=0}| + |[\varepsilon a u_{ttx}^2]_{x=1}| + 4C_t\varepsilon\int_0^1\left(-\frac{g_t}{g}\,q_xq_{xt} - \frac{d_t}{g}\,u_{tx}q_{tx} + \frac{\alpha_2}{g}\,q_t^2q_{tx} + \frac{\alpha_{2t}}{g}\,qq_tq_{tx} + \frac{\alpha_2}{g}\,qq_{tt}q_{tx}\right)\mathrm{d}x \tag{67}$$

Addition of (66) and (67) leads to

$$\left|\left[\frac{3g}{ad}(qu_{tx}+q_tu_{ttx})\right]_{x=0}^{x=1}\right|$$

$$\leqslant \frac{2C_t}{\varepsilon^3}(1+\varepsilon^2)\int_0^1 (q^2+q_t^2)\,\mathrm{d}x$$

$$+\int_0^1 \frac{36C_t\varepsilon}{a^2d^2}(\theta_{tt}^2+\theta_t^2)\,\mathrm{d}x+\int_0^1 \frac{36C_t\varepsilon}{a^2}(u_{ttx}^2+u_{tx}^2)\,\mathrm{d}x+\left|\left[\varepsilon a u_{ttx}^2+\varepsilon a u_{tx}^2\right]_{x=0}\right|$$

$$+\left|\left[\varepsilon a u_{ttx}^2+\varepsilon a u_{tx}^2\right]_{x=1}\right|+\int_0^1 \frac{2C_t\varepsilon a^2d^2}{9}(q_{tx}^2+q_x^2)\,\mathrm{d}x+\int_0^1 \frac{4C_t\varepsilon\alpha_2}{g}qq_tq_x\,\mathrm{d}x$$

$$+4C_t\varepsilon^2\int_0^1\left(-\frac{g_t}{g}q_xq_{xt}-\frac{d_t}{g}u_{tx}q_{tx}+\frac{\alpha_2}{g}q_t^2q_{tx}+\frac{\alpha_{2t}}{g}qq_tq_{tx}+\frac{\alpha_2}{g}qq_{tt}q_{tx}\right)\mathrm{d}x$$

Using (15) and Young's inequality we find

$$\int_0^1 \frac{2C_t\varepsilon a^2d^2}{9}q_x^2\,\mathrm{d}x \leqslant \int_0^1 \frac{4C_t\varepsilon a^2d^2}{9}\left(\frac{1}{g^2}\theta_t^2+\frac{d^2}{g^2}u_{tx}^2+\frac{\alpha_2}{g}qq_tq_x\right)\mathrm{d}x$$

By the same manner using Equation (30) we obtain

$$\int_0^1 \frac{2C_t\varepsilon a^2d^2}{9}q_{tx}^2\,\mathrm{d}x \leqslant \int_0^1 \frac{4C_t\varepsilon a^2d^2}{9}\left(\frac{1}{g^2}\theta_{tt}^2+\frac{d^2}{g^2}u_{ttx}^2-\frac{g_t}{g}q_xq_{tx}\right.$$

$$\left.-\frac{d_t}{g}u_{tx}q_{tx}+\frac{\alpha_2}{g}q_t^2q_{tx}+\frac{\alpha_{2t}}{g}q_tqq_{tx}+\frac{\alpha_2}{g}q_{tt}qq_{tx}\right)\mathrm{d}x$$

Then, we conclude

$$\left|\left[\frac{3g}{ad}(qu_{tx}+q_tu_{ttx})\right]_{x=0}^{x=1}\right|$$

$$\leqslant \frac{2C_t}{\varepsilon^3}(1+\varepsilon^2)\int_0^1 (q^2+q_t^2)\,\mathrm{d}x+\int_0^1 \frac{36C_t\varepsilon}{a^2d^2}(\theta_{tt}^2+\theta_t^2)\,\mathrm{d}x$$

$$+\int_0^1 \frac{36C_t\varepsilon}{a^2}(u_{ttx}^2+u_{tx}^2)\,\mathrm{d}x+|[\varepsilon a u_{ttx}^2+\varepsilon a u_{tx}^2]_{x=0}|+|[\varepsilon a u_{ttx}^2+\varepsilon a u_{tx}^2]_{x=1}|$$

$$+\int_0^1 \frac{4C_t\varepsilon a^2d^2}{9}\left(\frac{1}{g^2}\theta_t^2+\frac{d^2}{g^2}u_{tx}^2\right)\mathrm{d}x+\int_0^1 \frac{4C_t\varepsilon a^2d^2}{9}\left(\frac{1}{g^2}\theta_{tt}^2+\frac{d^2}{g^2}u_{ttx}^2\right)\mathrm{d}x+R_9 \quad (68)$$

where

$$R_9 = \int_0^1 \frac{4C_t\varepsilon\alpha_2}{g} qq_tq_x \,\mathrm{d}x + 4C_t\varepsilon^2 \int_0^1 \left(-\frac{g_t}{g} q_xq_{xt} - \frac{d_t}{g} u_{tx}q_{tx} + \frac{\alpha_2}{g} q_t^2 q_{tx} + \frac{\alpha_{2t}}{g} qq_tq_{tx} + \frac{\alpha_2}{g} qq_{tt}q_{tx}\right)\mathrm{d}x + \int_0^1 \frac{4C_t\varepsilon a^2d^2}{9}\left(\frac{\alpha_2}{g} qq_tq_x - \frac{g_t}{g} q_xq_{tx} - \frac{d_t}{g} u_{tx}q_{tx} + \frac{\alpha_2}{g} q_t^2q_{tx} + \frac{\alpha_{2t}}{g} q_tqq_{tx} + \frac{\alpha_2}{g} q_{tt}qq_{tx}\right)\mathrm{d}x \tag{69}$$

As in Reference [5], we multiply (29) by ϕu_{tx}, for $\phi(x) = 1 - 2x$, and integrate by parts to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{tt}\phi u_{tx}\,\mathrm{d}x + \frac{1}{2}(u_{tt}^2(1) + u_{tt}^2(0)) + \left[\frac{a}{2}u_{tx}^2\right]_{x=0} + \left[\frac{a}{2}u_{tx}^2\right]_{x=1} - \int_0^1 u_{tt}^2\,\mathrm{d}x + \int_0^1 (-au_{tx}^2 + b\theta_{tx}\phi u_{tx})\,\mathrm{d}x = \int_0^1 \left(-\frac{1}{2}a_x\phi u_{tx}^2 + a_tu_{xx}\phi u_{tx} - b_t\theta_x\phi u_{tx} + \alpha_1q_tq_x\phi u_{tx} + \alpha_1qq_{xt}\phi u_{tx} + \alpha_{1t}qq_x\phi u_{tx}\right)\mathrm{d}x \tag{70}$$

If we multiply (15) by $(-b/d)\phi\theta_{xt}$ and integrate by parts we get

$$\left[\frac{b}{2d}\theta_t^2\right]_{x=1} + \left[\frac{b}{2d}\theta_t^2\right]_{x=0} - \int_0^1 \frac{b}{d}\theta_t^2\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\frac{bg}{d}\phi q_x\theta_x\right)\mathrm{d}x + \int_0^1\left(\frac{bg}{d}\phi q_{xt}\theta_x - b\phi\theta_{xt}u_{tx}\right)\mathrm{d}x = -\int_0^1\left(\frac{\alpha_2 b}{d}\phi qq_t\theta_{tx} - \frac{1}{2}\left(\frac{b}{d}\right)_x\phi\theta_t^2 - \left(\frac{bg}{d}\right)_t\phi q_x\theta_x\right)\mathrm{d}x$$

which implies, using equation resulting from differentiation of (16) with respect to x,

$$\left[\frac{b}{2d}\theta_t^2\right]_{x=1} + \left[\frac{b}{2d}\theta_t^2\right]_{x=0} + \int_0^1\left(-\frac{b}{d}\theta_t^2\,\mathrm{d}x - \frac{bg}{d\tau}\phi\theta_xq_x - \frac{bgk}{d\tau}\theta_x^2 - bu_{tx}\phi\theta_{tx}\right)\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\frac{bg}{d}\phi q_x\theta_x\right)\mathrm{d}x + \left[\frac{bgk}{2d\tau}\theta_x^2\right]_{x=1} + \left[\frac{bgk}{2d\tau}\theta_x^2\right]_{x=0}$$

$$= \int_0^1 \left(-\frac{1}{2} \left(\frac{bg}{d\tau}\right)_x \phi\theta_x^2 + \frac{bg\tau_x}{d\tau}\,\phi\theta_x q_t + \frac{bgk_x}{d\tau}\,\phi\theta_x^2 \right) \mathrm{d}x$$
$$- \int_0^1 \left(\frac{\alpha_2 b}{d}\,\phi q q_t \theta_{tx} - \frac{1}{2}\left(\frac{b}{d}\right)_x \phi\theta_t^2 + \left(\frac{bg}{d}\right)_t \phi q_x \theta_x \right)\,\mathrm{d}x \tag{71}$$

Combining (70), (71) and using Young's inequality for $\int_0^1 (bg/d\tau)\phi\theta_x q_x\,\mathrm{d}x$, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(u_{tt}\phi u_{tx} - \frac{bg}{d}\,\phi q_x\theta_x\right)\mathrm{d}x + \left[\frac{a}{2}u_{tx}^2\right]_{x=0} + \left[\frac{a}{2}u_{tx}^2\right]_{x=1}$$
$$\leqslant \int_0^1 \left(u_{tt}^2 + a u_{tx}^2 + \frac{b}{d}\,\theta_t^2 + \left(\frac{bgk}{d\tau} + \frac{b^2 g^2}{4d^2\tau^2}\right)\theta_x^2 + q_x^2 \right)\,\mathrm{d}x + \Delta_1 \tag{72}$$

where

$$\Delta_1 = \int_0^1 \left(-\frac{1}{2}\,a_x\phi u_{tx}^2 + a_t u_{xx}\phi u_{tx} - b_t\theta_x\phi u_{tx} + \alpha_1 q_t q_x \phi u_{tx} + \alpha_1 q q_{xt}\phi u_{tx} \right.$$
$$+ \alpha_1 q q_x \phi u_{tx} - \frac{1}{2}\left(\frac{bg}{d\tau}\right)_x \phi\theta_x^2 + \frac{bg\tau_x}{d\tau}\,\phi\theta_x q_t + \frac{bgk_x}{d\tau}\,\phi\theta_x^2$$
$$\left. - \frac{\alpha_2 b}{d}\,\phi q q_t\theta_{tx} - \frac{1}{2}\left(\frac{b}{d}\right)_x \phi\theta_t^2 - \left(\frac{bg}{d}\right)_t \phi\theta_x q_x \right)\mathrm{d}x \tag{73}$$

Next, we multiply (15) by q_x and use Young's inequality to get

$$\int_0^1 q_x^2\,\mathrm{d}x \leqslant \int_0^1 \left(\frac{2}{g^2}\,\theta_t^2 + \frac{2d^2}{g^2}\,u_{tx}^2 + \frac{2\alpha_2}{g}\,q q_t q_x \right)\mathrm{d}x$$

Thus (72) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(u_{tt}\phi u_{tx} - \frac{bg}{d}\,\phi q_x\theta_x \right)\,\mathrm{d}x + \left[\frac{a}{2}\,u_{tx}^2\right]_{x=0} + \left[\frac{a}{2}\,u_{tx}^2\right]_{x=1}$$
$$\leqslant \int_0^1 \left(u_{tt}^2 + \left(a + \frac{2d^2}{g^2}\right) u_{tx}^2 + \left(\frac{b}{d} + \frac{2}{g^2}\right)\theta_t^2 + \left(\frac{bgk}{d\tau} + \frac{b^2g^2}{4d^2\tau^2}\right)\theta_x^2 \right.$$
$$\left. + \frac{2\alpha_2}{g} q q_t q_x \right)\mathrm{d}x + \Delta_1 \tag{74}$$

Similarly, multiplying (33) by ϕu_{ttx}, we obtain, after some manipulations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{ttt}\phi u_{ttx}\,\mathrm{d}x + \left[\frac{a}{2}u_{ttx}^2\right]_{x=0} + \left[\frac{a}{2}u_{ttx}^2\right]_{x=1} + \frac{1}{2}(u_{ttt}^2(1) + u_{ttt}^2(0))$$
$$+ \int_0^1 (-au_{ttx}^2 - u_{ttt}^2 + b\theta_{xtt}\phi u_{ttx})\,\mathrm{d}x = \Delta_2 + \frac{1}{2}\int_0^1 a_x\phi u_{ttx}^2\,\mathrm{d}x \quad (75)$$

where

$$\Delta_2 = \int_0^1 [2a_t u_{xxt}\phi u_{ttx} - 2b_t\theta_{xt}\phi u_{ttx} + a_{tt}u_{xx}\phi u_{ttx} - b_{tt}\theta_x\phi u_{ttx} + \phi u_{ttx}(\alpha_1 q_{tt}q_x$$
$$+ 2\alpha_1 q_t q_{tx} + 2\alpha_{1t}q_t q_x + \alpha_1 q q_{ttx} + 2\alpha_{1t}q q_{tx} + \alpha_{1tt}q q_x)]\,\mathrm{d}x$$

The multiplication of (30) by $-(b/d)\phi\theta_{ttx}$ yields, by similar computations,

$$\int_0^1 -\frac{b}{d}\theta_{tt}^2\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{bg}{d}\phi\theta_{tx}q_{xt}\,\mathrm{d}x + \left[\frac{b}{2d}\theta_{tt}^2\right]_{x=1} + \left[\frac{b}{2d}\theta_{tt}^2\right]_{x=0}$$
$$+ \int_0^1 \left(\frac{bg}{d}q_{ttx}\phi\theta_{tx} - b\phi u_{ttx}\theta_{ttx}\right)\mathrm{d}x = -\frac{1}{2}\int_0^1 \left(\frac{b}{d}\right)_x \phi\theta_{tt}^2\,\mathrm{d}x$$
$$- \int_0^1 \left(\frac{bg}{d}\right)_t q_{tx}\phi\theta_{tx}\,\mathrm{d}x - \int_0^1 \frac{b}{d}\phi u_{ttx}(-g_t q_x - d_t u_{tx} + \alpha_2 q_t^2 + \alpha_{2t}q_t q + \alpha_2 q_{tt}q)\,\mathrm{d}x$$

which implies, using equation resulting from differentiation of (31) with respect to x and integration by parts,

$$\int_0^1 -\frac{b}{d}\theta_{tt}^2\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{bg}{d}\phi\theta_{tx}q_{xt}\,\mathrm{d}x + \left[\frac{b}{2d}\theta_{tt}^2\right]_{x=1} + \left[\frac{b}{2d}\theta_{tt}^2\right]_{x=0}$$
$$\int_0^1 \left(-b\phi\theta_{ttx}u_{ttx} - \frac{bg}{d\tau}\phi\theta_{tx}q_{xt} - \frac{bgk}{d\tau}\theta_{tx}^2\right)\mathrm{d}x + \left[\frac{bgk}{2d\tau}\theta_{tx}^2\right]_{x=1} + \left[\frac{bgk}{2d\tau}\theta_{tx}^2\right]_{x=0} = \Delta_3 \quad (76)$$

where

$$\Delta_3 = \int_0^1 \left\{\frac{1}{2}\left(\frac{bg}{d\tau}\right)_x \phi\theta_{tx}^2 + \frac{bg\tau_t}{d\tau}\phi\theta_{tx}q_{tx} + \frac{bgk_t}{d\tau}\phi\theta_{tx}\theta_{xx} + \frac{bg\tau_x}{d\tau}\phi\theta_{tx}q_{tt}\right.$$
$$\left. + \frac{bg\tau_{xt}}{d\tau}\phi\theta_{tx}q_t + \frac{bgk_x}{d\tau}\phi\theta_{tx}^2 + \frac{bgk_{tx}}{d\tau}\phi\theta_x\theta_{tx} - \frac{1}{2}\left(\frac{b}{d}\right)_x\theta_{tt}^2 - \left(\frac{bg}{d}\right)_t q_{tx}\phi\theta_{tx}\right\}\mathrm{d}x$$
$$+ \int_0^1 -\frac{b}{d}\phi u_{ttx}(-g_t q_x - d_t u_{tx} + \alpha_2 q_t^2 + \alpha_{2t}q_t q + \alpha_2 q_{tt}q)\,\mathrm{d}x$$

Combining (75) and (76) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(u_{ttt}\phi u_{ttx}-\frac{bg}{d}\,\phi\theta_{tx}q_{xt}\right)\,\mathrm{d}x+\left[\frac{a}{2}\,u_{ttx}^2\right]_{x=0}+\left[\frac{a}{2}\,u_{ttx}^2\right]_{x=1}-(\Delta_2+\Delta_3)$$
$$\leqslant\int_0^1\left(u_{ttt}^2+au_{ttx}^2+\frac{b}{d}\,\theta_{tt}^2+\frac{bgk}{d\tau}\,\theta_{tx}^2+\frac{b^2g^2}{4d^2\tau^2}\,\theta_{tx}^2+q_{tx}^2\right)\mathrm{d}x+\frac{1}{2}\int_0^1 a_x\phi u_{ttx}^2\,\mathrm{d}x \qquad (77)$$

Also multiplying (30) by $-(1/g)q_{tx}$ and exploiting Young's inequality gives

$$\int_0^1 q_{tx}^2\,\mathrm{d}x\leqslant\int_0^1\left(\frac{2}{g^2}\,\theta_{tt}^2+\frac{2d^2}{g^2}u_{ttx}^2\right)\mathrm{d}x$$
$$+\int_0^1 2\left(-\frac{g_t}{g}q_xq_{tx}-\frac{d_t}{g}u_{tx}q_{tx}+\frac{\alpha_2}{g}\,q_t^2q_{tx}+\frac{\alpha_{2t}}{g}\,q_tqq_{tx}+\frac{\alpha_2}{g}\,q_{tt}qq_{tx}\right)\mathrm{d}x$$

Therefore (77) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(u_{ttt}\phi u_{ttx}-\frac{bg}{d}\,\phi\theta_{tx}q_{xt}\right)\mathrm{d}x+\left[\frac{a}{2}u_{ttx}^2\right]_{x=0}+\left[\frac{a}{2}u_{ttx}^2\right]_{x=1}$$
$$\leqslant\int_0^1\left(u_{ttt}^2+\left(a+\frac{2d^2}{g^2}\right)u_{ttx}^2+\left(\frac{b}{d}+\frac{2}{g^2}\right)\theta_{tt}^2+\left(\frac{bgk}{d\tau}+\frac{b^2g^2}{4d^2\tau^2}\right)\theta_{tx}^2\right)\mathrm{d}x+\Delta_4 \quad (78)$$

where

$$\Delta_4=\Delta_2+\Delta_3+\int_0^1\left(\frac{1}{2}\,a_x\phi u_{ttx}^2-\frac{2g_t}{g}\,q_xq_{tx}-\frac{2d_t}{g}\,u_{tx}q_{tx}\right.$$
$$\left.+\frac{2\alpha_2}{g}\,q_t^2q_{tx}+\frac{2\alpha_{2t}}{g}\,q_tqq_{tx}+\frac{2\alpha_2}{g}\,q_{tt}qq_{tx}\right)\mathrm{d}x$$

Adding (74) and (78) we obtain, after multiplication by 2ε,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 2\varepsilon(u_{tt}\phi u_{tx}+u_{ttt}\phi u_{ttx}-\frac{bg}{d}\,(\phi\theta_xq_x+\phi\theta_{tx}q_{xt}))\,\mathrm{d}x$$
$$+\varepsilon[a(u_{ttx}^2+u_{tx}^2)]_{x=0}+\varepsilon[a(u_{ttx}^2+u_{tx}^2)]_{x=1}$$
$$\leqslant 2\varepsilon\int_0^1(u_{tt}^2+u_{ttt}^2)\,\mathrm{d}x+2\varepsilon\int_0^1\left(a+2\,\frac{d^2}{g^2}\right)(u_{tx}^2+u_{ttx}^2)\,\mathrm{d}x$$

$$+2\varepsilon\int_0^1\left(\frac{b}{d}+\frac{2}{g^2}\right)(\theta_t^2+\theta_{tt}^2)\,\mathrm{d}x$$

$$+2\varepsilon\int_0^1\left(\frac{bgk}{d\tau}+\frac{b^2g^2}{4d^2\tau^2}\right)(\theta_{tx}^2+\theta_x^2)\,\mathrm{d}x+R_{10} \tag{79}$$

where

$$R_{10}=\int_0^1\frac{2\varepsilon\alpha_2}{g}\,qq_tq_{tx}+2\varepsilon(\Delta_1+\Delta_4) \tag{80}$$

A combination of (68) and (79) yields

$$\left|\left[\frac{3g}{ad}(qu_{tx}+q_tu_{ttx})\right]_{x=0}^{x=1}\right|$$
$$\leqslant\frac{2C_t}{\varepsilon^3}(1+\varepsilon^2)\int_0^1(q^2+q_t^2)\,\mathrm{d}x+\int_0^1\frac{36C_t\varepsilon}{a^2d^2}(\theta_{tt}^2+\theta_t^2)\,\mathrm{d}x$$
$$+\int_0^1\frac{36C_t\varepsilon}{a^2}(u_{ttx}^2+u_{tx}^2)\,\mathrm{d}x+\int_0^1\frac{4C_t\varepsilon a^2d^2}{9}\left(\frac{1}{g^2}\theta_t^2+\frac{d^2}{g^2}u_{tx}^2\right)\mathrm{d}x$$
$$+\int_0^1\frac{4C_t\varepsilon a^2d^2}{9}\left(\frac{1}{g^2}\theta_{tt}^2+\frac{d^2}{g^2}u_{ttx}^2\right)\mathrm{d}x-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 2\varepsilon\left(u_{tt}\phi u_{tx}\right.$$
$$\left.+u_{ttt}\phi u_{ttx}-\frac{bg}{d}(\phi\theta_xq_x+\phi\theta_{tx}q_{xt})\right)\,\mathrm{d}x+2\varepsilon\int_0^1(u_{tt}^2+u_{ttt}^2)\,\mathrm{d}x$$
$$+2\varepsilon\int_0^1\left(a+2\frac{d^2}{g^2}\right)(u_{tx}^2+u_{ttx}^2)\,\mathrm{d}x+2\varepsilon\int_0^1\left(\frac{b}{d}+\frac{2}{g^2}\right)(\theta_t^2+\theta_{tt}^2)\mathrm{d}x$$
$$+2\varepsilon\int_0^1\left(\frac{bgk}{d\tau}+\frac{b^2g^2}{4d^2\tau^2}\right)(\theta_x^2+\theta_{tx}^2)\,\mathrm{d}x+R_9+R_{10} \tag{81}$$

Combining (46), (57), (81) and using (19) and (20) we obtain for sufficiently small ε:

$$\int_0^1\frac{2}{a}(u_{tx}^2+u_{ttx}^2)\,\mathrm{d}x+\int_0^1\frac{1}{2}(u_{xx}^2+u_{xxt}^2)\,\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}G_1(t)$$
$$\leqslant\int_0^1\left(\frac{3b^2}{4a^2}+\frac{3b}{ad}+\frac{27}{d^2}\right)(\theta_x^2+\theta_{tx}^2)\,\mathrm{d}x+\int_0^1\frac{27g^2}{a^2d^2}(q_t^2+q_{tt}^2)\,\mathrm{d}x$$
$$+\frac{2C_t}{\varepsilon^3}(1+\varepsilon^2)\int_0^1(q^2+q_t^2)\,\mathrm{d}x+\int_0^1\frac{36C_t\varepsilon}{a^2d^2}(\theta_{tt}^2+\theta_t^2)\,\mathrm{d}x$$

$$
\begin{aligned}
&+\int_0^1 \frac{4C_t\varepsilon a^2 d^2}{9}\left(\frac{1}{g^2}\,\theta_t^2+\frac{d^2}{g^2}\,u_{tx}^2\right)\mathrm{d}x+\int_0^1 \frac{4C_t\varepsilon a^2 d^2}{9}\left(\frac{1}{g^2}\,\theta_{tt}^2+\frac{d^2}{g^2}\,u_{ttx}^2\right)\mathrm{d}x\\
&+\int_0^1 \frac{36C_t\varepsilon}{a^2}\,(u_{ttx}^2+u_{tx}^2)\,\mathrm{d}x+2\varepsilon\int_0^1 (u_{tt}^2+u_{ttt}^2)\,\mathrm{d}x\\
&+2\varepsilon\int_0^1\left(a+2\,\frac{d^2}{g^2}\right)(u_{tx}^2+u_{ttx}^2)\,\mathrm{d}x+2\varepsilon\int_0^1\left(\frac{b}{d}+\frac{2}{g^2}\right)(\theta_t^2+\theta_{tt}^2)\,\mathrm{d}x\\
&+2\varepsilon\int_0^1\left(\frac{bgk}{d\tau}+\frac{b^2g^2}{4d^2\tau^2}\right)(\theta_x^2+\theta_{tx}^2)\,\mathrm{d}x+R_4+R_5+R_9+R_{10}
\end{aligned}
$$

where

$$
\begin{aligned}
G_1(t)=\int_0^1 &\left[\frac{1}{a}\,(u_{tx}u_x+u_{ttx}u_{tx})-\frac{3g}{a^2d}\,(qu_{tt}+q_tu_{ttt})+\frac{3gb\tau}{a^2dk}\,(qq_t+q_tq_{tt})\right.\\
&+\frac{3bg}{a^2dk}\,(q^2+q_t^2)+\frac{3\tau}{adk}\,(u_tq_t+u_{tt}q_{tt})+\frac{3}{adk}\,(u_tq+u_{tt}q_t)\\
&-\frac{3\alpha_1}{a^2d}\,q_tq\theta_{tt}-\frac{3\alpha_1}{a^2}\,q_tqu_{ttx}-\frac{3\alpha_1\alpha_2}{a^2d}\,q_tq^2q_{tt}\\
&\left.+2\varepsilon(u_{tt}\phi u_{tx}+u_{ttt}\phi u_{ttx}-\frac{bg}{d}\,(\phi\theta_xq_x+\phi\theta_{tx}q_{xt})\right]\,\mathrm{d}x
\end{aligned}
$$

Then we have, by (19) and (20),

$$
\begin{aligned}
&c_1\int_0^1 (u_{tx}^2+u_{ttx}^2)\,\mathrm{d}x+c_2\int_0^1 (u_{xx}^2+u_{xxt}^2)\,\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\,G_1(t)\\
&\leqslant c_3\int_0^1 (\theta_x^2+\theta_{tx}^2)\,\mathrm{d}x+c_4\int_0^1 (q_t^2+q_{tt}^2)\,\mathrm{d}x+\frac{c_5}{\varepsilon^3}\int_0^1 (q^2+q_t^2)\,\mathrm{d}x\\
&\quad+c_6\varepsilon\int_0^1 (\theta_{tt}^2+\theta_t^2)\,\mathrm{d}x+c_7\varepsilon\int_0^1 (u_{ttx}^2+u_{tx}^2)\,\mathrm{d}x\\
&\quad+c_8\varepsilon\int_0^1 (u_{tt}^2+u_{ttt}^2)\,\mathrm{d}x+c_9\varepsilon\int_0^1 (\theta_x^2+\theta_{tx}^2)\,\mathrm{d}x+R_4+R_5+R_9+R_{10} \qquad (82)
\end{aligned}
$$

Adding σ (64) to (82) we get

$$
c_1\int_0^1 (u_{tx}^2+u_{ttx}^2)\,\mathrm{d}x+c_2\int_0^1 (u_{xx}^2+u_{xxt}^2)\,\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\,G_1(t)+\sigma\int_0^1 (\theta_{tt}^2+\theta_t^2)\,\mathrm{d}x
$$

$$
\begin{aligned}
&-\sigma \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 2g(q\theta_x + q_t\theta_{xt})\,\mathrm{d}x \\
&\leqslant c_3\int_0^1(\theta_x^2+\theta_{tx}^2)\,\mathrm{d}x + c_4\int_0^1(q_t^2+q_{tt}^2)\,\mathrm{d}x + \frac{c_5}{\varepsilon^3}\int_0^1(q^2+q_t^2)\,\mathrm{d}x \\
&\quad + c_6\varepsilon\int_0^1(\theta_{tt}^2+\theta_t^2)\,\mathrm{d}x + c_7\varepsilon\int_0^1(u_{ttx}^2+u_{tx}^2)\,\mathrm{d}x + c_8\varepsilon\int_0^1(u_{tt}^2+u_{ttt}^2)\,\mathrm{d}x \\
&\quad + c_9\varepsilon\int_0^1(\theta_x^2+\theta_{tx}^2)\,\mathrm{d}x + c_{10}\int_0^1(q_t^2+q_{tt}^2+\theta_x^2+\theta_{tx}^2)\,\mathrm{d}x + \sigma\int_0^1 d^2(u_{tx}^2+u_{ttx}^2)\,\mathrm{d}x \\
&\quad + R_4 + R_5 + c_{10}R_8 + R_9 + R_{10}
\end{aligned}
$$

where c_1–c_{10} are constants depending on β and the upper of the functions a, b, g, d, τ, k, α_1, α_2 and their derivatives only. Using (43), (57), (63), (81) and choosing ε small and σ such that $c_1 - \sigma\|d\|_\infty^2 > 0$, we have

$$
\begin{aligned}
&c_{11}\int_0^1(u_{tx}^2+u_{ttx}^2)\,\mathrm{d}x + c_{12}\int_0^1(u_{xx}^2+u_{xxt}^2)\,\mathrm{d}x + c_{13}\int_0^1(\theta_{tt}^2+\theta_t^2)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}G_2(t) \\
&\leqslant c_{14}\int_0^1(q^2+q_t^2+q_{tt}^2)\,\mathrm{d}x + c_{15}(R_4+R_5+R_8+R_9+R_{10}+\widetilde{R_1})
\end{aligned}
\tag{83}
$$

where

$$
G_2(t) = G_1(t) - \sigma\int_0^1 2g(q\theta_x + q_t\theta_{xt})\,\mathrm{d}x
$$

For $\gamma > 0$ we define the Lyaponov function

$$
F(t) = \frac{1}{\gamma}E(t) + G_2(t) \tag{84}
$$

Lemma 3.1
There exists a constant $c_{16} > 0$ such that

$$
\left(\frac{1}{\gamma} - c_{16}\{1+\alpha(t)+\alpha^2(t)\}\right)E(t) \leqslant F(t) \leqslant \left(\frac{1}{\gamma} + c_{16}\{1+\alpha(t)+\alpha^2(t)\}\right)E(t) \tag{85}
$$

Proof
It suffices to show, for a constant $c_{16} > 0$, that

$$
|G_2(t)| \leqslant c_{16}(1+\alpha(t)+\alpha^2(t))E(t) \tag{86}
$$

To do this, we estimate all the terms of $G_2(t)$, making use of Young's inequality, Schwarz inequality, and Poincare's inequality as in, for instance,

$$
\int_0^1 \frac{3\alpha_1\alpha_2}{a^2 d} q_t q^2 q_{tt}\,\mathrm{d}x \leqslant 3\left|\left|\frac{\alpha_1\alpha_2}{a^2 d}\right|\right|_\infty \|q^2(.,t)\|_\infty \int_0^1 |q_t q_{tt}|\,\mathrm{d}x \leqslant C\alpha^2(t)E(t)
$$

where C depends only on $a, b, g, d, \tau, k, \alpha_1, \alpha_2$ and their derivatives. Hence, by carrying all calculations (86) is established hence the assertion of the lemma is proved. □

Lemma 3.2
There exist $c_{17}, c_{18} > 0$, such that

$$c_{17}E(t) \leqslant \Lambda(t) \leqslant c_{18}\,(1 + \alpha(t) + \alpha^2(t) + \alpha^3(t) + \alpha^4(t))E(t) \tag{87}$$

Proof
The first inequality in (87) is obvious, the second is established by estimating all the terms as in Lemma 3.1. For instance we multiply (30) by q_{xt} to get

$$\int_0^1 q_{tx}^2\,\mathrm{d}x = \int_0^1 \frac{q_{xt}}{g}\,(-\theta_{tt} - g_t q_x - d u_{ttx} - d_t u_{tx} + \alpha_2 q_t^2 + \alpha_{2t} q_t q + \alpha_2 q_{tt} q)\,\mathrm{d}x$$

and by the same manner, using (19), (20), and Young's inequality, we obtain

$$\begin{aligned}\int_0^1 q_{tx}^2\,\mathrm{d}x &\leqslant C\left(\int_0^1 \theta_{tt}^2 + \alpha(t) q_x^2 + u_{ttx}^2 + \alpha(t) u_{tx}^2 + \alpha(t) q_t^2 + \alpha^2(t) q^2 + \alpha(t) q_{tt}^2\right)\\ &\leqslant C(1 + \alpha(t) + \alpha^2(t))E(t)\end{aligned}$$

By carrying all the calculations, we arrive at

$$\Lambda(t) \leqslant C_{18}(1 + \alpha(t) + \alpha^2(t) + \alpha^3(t) + \alpha^4(t))E(t) \tag{88}$$

This completes the proof of Lemma 3.2.

Next using (26), (32), (36), (43), (60), (61), and (83) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) \leqslant -c_{19}E(t) + |R_1 + R_2 + \cdots + R_{10} + \widetilde{R_1}| \tag{89}$$

Lemma 3.3

$$|R_1 + R_2 + \cdots + R_{10} + \widetilde{R_1}| \leqslant c_{20}(\alpha(t) + \alpha^2(t) + \alpha^3(t))\Lambda(t) \tag{90}$$

Proof
The proof is also similar to the one of Lemma 3.1. We only consider those of highest order; namely $\int_0^1 \alpha_{1tt} k d q q_x u_{ttt}\,\mathrm{d}x$, $\int_0^1 \alpha_{2tt} k b q q_t \theta_{tt}\,\mathrm{d}x$. We first compute

$$\alpha_{1tt} = \alpha_{1u_xu_x} u_{xt}^2 + \alpha_{1u_x} u_{xtt} + 2\alpha_{u_x\theta}\theta_t u_{xt} + \alpha_{1\theta\theta}\theta_t^2 + \alpha_{1u_x}\theta_{tt}$$

By using the fact that $\alpha_1, k, d \in C_b^3$, we obtain

$$\left|\int_0^1 \alpha_{1tt} k d q q_x u_{ttt}\,\mathrm{d}x\right| \leqslant C\left\{\int_0^1 |q q_x u_{ttt} u_{xt}^2|\,\mathrm{d}x + \int_0^1 |q q_x u_{ttt} u_{xt}|\,\mathrm{d}x\right.$$

$$+\int_0^1 |qq_x u_{ttt} u_{xt}\theta_t|\,\mathrm{d}x + \int_0^1 |qq_x u_{ttt}\theta_t^2|\,\mathrm{d}x + \int_0^1 |qq_x u_{ttt}\theta_{tt}|\,\mathrm{d}x \Bigg\}$$

$$\leqslant C\{\alpha^3(t)+\alpha^2(t)\}E(t)$$

Similarly the other integral is treated. By carrying all calculations, the proof of the lemma is completed. □

By combining (85), (87), and (89) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) \leqslant -\gamma c_{19}F(t) + \frac{c_{18}(1+\alpha(t)+\alpha^2(t)+\alpha^3(t)+\alpha^4(t))(\alpha(t)+\alpha^2(t)+\alpha^3(t))}{(1/\gamma)-c_{16}(1+\alpha(t)+\alpha^2(t))}F(t) \tag{91}$$

At this point, we choose γ so that $(1/\gamma)>2c_{16}$. Once is fixed we pick δ, in (25), small that enough so that

$$+\frac{c_{18}(1+\alpha(0)+\alpha^2(0)+\alpha^3(0)+\alpha^4(0))(\alpha(0)+\alpha^2(0)+\alpha^3(0))}{(1/\gamma)-c_{16}(1+\alpha(0)+\alpha^2(0))} \leqslant \frac{1}{4}\gamma c_{19}$$

hence (91) yields, for some $t_0>0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) \leqslant -c_5F(t) \quad \forall t\in[0,t_0)$$

Direct integration then leads to

$$F(t)\leqslant F(0)e^{-c_{17}t} \quad \forall t\in[0,t_0) \tag{92}$$

Since $F(t)\leqslant F(0)$ we extend (92) beyond t_0. By repeating the same procedure, taking δ even smaller if necessary, and using the continuity of F, (92) is established for all $t\geqslant 0$. This completes the proof of the theorem. □

Remark 3.1

The proof shows that the initial data can be taken in a neighbourhood of the equilibrium state $(0,0,0)$, in which the solution remains for ever. Therefore the result is also valid for $a,b,g,d,\tau,k,\alpha_1,\alpha_2$ in C^3 instead in C_b^3 as mentioned in Remark 2.1.

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