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Global non-existence of solutions of a class of wave equations with non-linear damping and source terms

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SUMMARY

In this paper we consider the non-linear wave equation

$$u_{tt} - \Delta u_t - \operatorname{div}(|\nabla u|^{\alpha-2}\nabla u) - \operatorname{div}(|\nabla u_t|^{\beta-2}\nabla u_t) + a|u_t|^{m-2}u_t = b|u|^{p-2}u$$

$a, b > 0$, associated with initial and Dirichlet boundary conditions. We prove, under suitable conditions on α, β, m, p and for negative initial energy, a global non-existence theorem. This improves a result by Yang (*Math. Meth. Appl. Sci.* 2002; **25**:825–833), who requires that the initial energy be sufficiently negative and relates the global non-existence of solutions to the size of Ω. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: non-linear damping; non-linear source; negative initial energy; global non-existence

1. INTRODUCTION

In this paper we are concerned with the following initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u_t - \operatorname{div}(|\nabla u|^{\alpha-2}\nabla u) - \operatorname{div}(|\nabla u_t|^{\beta-2}\nabla u_t) \\ \qquad +a|u_t|^{m-2}u_t = b|u|^{p-2}u, \quad x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0 \end{cases} \tag{1}$$

where $a, b > 0$, $\alpha, \beta, m, p > 2$, and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$, with a smooth boundary $\partial\Omega$.

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Equation (1) appears in the models of non-linear viscoelasticity (see References [1–3]). It also can be considered as a system governing the longitudinal motion of a viscoelastic configuration obeying a non-linear Voight model (see References [3,4]).

In the absence of viscosity and strong damping, Equation (1) becomes

$$u_{tt} - \operatorname{div}(|\nabla u|^{\alpha-2}\nabla u) + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \qquad x \in \Omega, \ t > 0 \tag{2}$$

For $b = 0$, it is well known that the damping term assures global existence and decay of the solution energy for arbitrary initial data (see References [5,6]). Then, for $a = 0$ the source term causes finite time blow up of solutions with negative initial energy if $p > \alpha$ (see References [7,8]).

The interaction between the damping and the source terms was first considered by Levine [9,10] in the linear damping case ($\alpha = m = 2$). He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [11] extended Levine's result to the non-linear damping case ($m > 2$). In their work, the authors considered (2) with $\alpha = 2$ and introduced a method different than the one known as the concavity method. They determined suitable relations between m and p, for which there is global existence or alternatively finite time blow up. Precisely; they showed that solutions with negative energy continue to exist globally 'in time' if $m \geqslant p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative. This result was later generalized to an abstract setting and to unbounded domains by Levine and Serrin [12] and Levine *et al.* [13]. In these papers, the authors showed that no solution with negative energy can be extended on $[0, \infty)$ if $p > m$ and proved several non-continuation theorems. This generalization allowed them also to apply their result to quasilinear situations ($\alpha > 2$), of which the problem in Reference [11] is a particular case. Vitillaro [14] combined the arguments in References [11,12] to extend these results to situations where the damping is non-linear and the solution has positive initial energy. Similar results have also been established by Todorova [15,16] for different Cauchy problems.

In Reference [3], Yang studied (1) and proved a blow up result under the condition $p > \max\{\alpha, m\}$, $\alpha > \beta$, and the initial energy is sufficiently negative (see condition (ii) Theorem 2.1 of Reference [3]). In fact this condition made it clear that there exists a certain relation between the blow-up time and $|\Omega|$ (see Remark 2 of Reference [3]). We should note here that (1) corresponds to Equation (5) of [3] but the same conclusions hold for Equation (1) of the same paper, under suitable conditions, stated in Theorem 2.3 of [3].

In this work we show that any weak solution of (1), with negative initial energy, cannot exists for all time if $p > \max\{\alpha, m\}$, $\alpha > \beta$. Therefore, our result improves the one of [3]. Our technique of proof follows closely the argument of [17] with the modifications needed for our problem.

2. BLOW UP

In order to state and prove our result, we introduce the following function space

$$Z = L^{\infty}([0, T); W_0^{1,\alpha}(\Omega)) \cap W^{1,\infty}([0, T); L^2(\Omega))$$
$$\cap\, W^{1,\beta}([0, T); W_0^{1,\beta}(\Omega)) \cap W^{1,m}([0, T); L^m(\Omega))$$

for $T>0$ and the energy functional

$$E(t)=\frac{1}{2}\int_{\Omega}u_t^2\,dx+\frac{1}{\alpha}\int_{\Omega}|\nabla u|^{\alpha}\,dx-\frac{b}{p}\int_{\Omega}|u|^p\,dx \tag{3}$$

Theorem
Assume that $\alpha,\beta,m,p\geqslant 2$ such that $\beta<\alpha$, and $\max\{m,\alpha\}<p<r_\alpha$, where r_α is the Sobolev critical exponent of $W_0^{1,\alpha}(\Omega)$. Assume further that

$$E(0)<0 \tag{4}$$

Then the solution $u\in Z$, of (1), cannot exist for all time.

Remark 2.1
We remind that $r_\alpha=n\alpha/(n-\alpha)$, if $n>\alpha$, $r_\alpha>\alpha$ if $n=\alpha$, and $r_\alpha=\infty$ if $n<\alpha$.

Remark 2.2
If the solution u is smooth enough then it blows up in finite time.

Proof
We suppose that the solution exists for all time and we reach to a contradiction. For this purpose we multiply Equation (1) by u_t and integrate over Ω to obtain

$$E'(t)=-\int_{\Omega}|\nabla u_t|^2\,dx-\int_{\Omega}|\nabla u_t|^{\beta}\,dx-a\int_{\Omega}|u_t|^m\,dx\leqslant 0 \tag{5}$$

for any regular solution. This remains valid for $u\in Z$ by density argument. Hence $E(t)\leqslant E(0)$, $\forall t\geqslant 0$.

By setting $H(t)=-E(t)$, we get

$$0<H(0)\leqslant H(t)\leqslant\frac{b}{p}\int_{\Omega}|u|^p\,dx,\quad \forall t\geqslant 0 \tag{6}$$

We then define

$$L(t)=H^{1-\sigma}(t)+\varepsilon\int_{\Omega}uu_t\,dx \tag{7}$$

for ε small to be chosen later and

$$0<\sigma\leqslant\min\left(\frac{\alpha-2}{p},\frac{\alpha-\beta}{p(\beta-1)},\frac{p-m}{p(m-1)},\frac{\alpha-2}{2\alpha}\right) \tag{8}$$

Our goal is to show that $L(t)$ satisfies a differential inequality of the form

$$L'(t)\geqslant\xi L^q(t),\quad q>1$$

This, of course, will lead to a blow up in finite time.
By taking a derivative of (7) we obtain

$$L'(t)=(1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\int_{\Omega}u_t^2\,dx+\varepsilon\int_{\Omega}uu_{tt}\,dx \tag{9}$$

By using Equation (1), the estimate (9) gives

$$\begin{aligned} L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon \int_\Omega u_t^2 \,\mathrm{d}x \\ - \varepsilon \int_\Omega \nabla u \nabla u_t \,\mathrm{d}x - \varepsilon \int_\Omega |\nabla u|^\alpha \,\mathrm{d}x \\ - \varepsilon \int_\Omega |\nabla u_t|^{\beta-2} \nabla u_t \nabla u \,\mathrm{d}x \\ - a\varepsilon \int_\Omega |u_t|^{m-2} u_t u \,\mathrm{d}x + b\varepsilon \int_\Omega |u|^p \,\mathrm{d}x \end{aligned} \tag{10}$$

We then exploit Young's inequality to get

$$\int_\Omega |u_t|^{m-2} u_t u \,\mathrm{d}x \leqslant \frac{\delta^m}{m} \int_\Omega |u|^m \,\mathrm{d}x + \frac{m-1}{m} \delta^{-m/(m-1)} \int_\Omega |u_t|^m \,\mathrm{d}x \tag{11}$$

$$\int_\Omega \nabla u \nabla u_t \,\mathrm{d}x \leqslant \frac{1}{4\mu} \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \mu \int_\Omega |\nabla u_t|^2 \,\mathrm{d}x \tag{12}$$

$$\int_\Omega |\nabla u_t|^{\beta-1} \nabla u \,\mathrm{d}x \leqslant \frac{\lambda^\beta}{\beta-1} \int_\Omega |\nabla u|^\beta \,\mathrm{d}x + \frac{\beta-1}{\beta} \lambda^{-\beta/(\beta-1)} \int_\Omega |\nabla u_t|^\beta \,\mathrm{d}x \tag{13}$$

A substitution of (11)–(13) in (10) yields

$$\begin{aligned} L'(t) \geqslant (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon \int_\Omega u_t^2 \,\mathrm{d}x \\ - \frac{\varepsilon}{4\mu} \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \mu\varepsilon \int_\Omega |\nabla u_t|^2 \,\mathrm{d}x \\ - \varepsilon \int_\Omega |\nabla u|^\alpha \,\mathrm{d}x - \varepsilon \frac{\lambda^\beta}{\beta} \int_\Omega |\nabla u|^\beta \,\mathrm{d}x \\ - \varepsilon \frac{\beta-1}{\beta} \lambda^{-\beta/(\beta-1)} \int_\Omega |\nabla u_t|^\beta \,\mathrm{d}x \\ + b\varepsilon \int_\Omega |u|^p \,\mathrm{d}x - a\varepsilon \frac{\delta^m}{m} \int_\Omega |u|^m \,\mathrm{d}x \\ - a\varepsilon \frac{m-1}{m} \delta^{-m/(m-1)} \int_\Omega |u_t|^m \,\mathrm{d}x \end{aligned} \tag{14}$$

Therefore by choosing δ, μ, λ so that

$$\begin{cases} \delta^{-m/(m-1)} = M_1 H^{-\sigma}(t) \\ \mu = M_2 H^{-\sigma}(t) \\ \lambda^{-\beta/(\beta-1)} = M_3 H^{-\sigma}(t) \end{cases}$$

for M_1, M_2, and M_3 to be specified later, and using (14) we arrive at

$$\begin{aligned} L'(t) \geqslant{} & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \int_\Omega u_t^2 \, dx \\ & - \frac{\varepsilon}{4M_2} H^\sigma(t) \int_\Omega |\nabla u|^2 \, dx - \varepsilon \int_\Omega |\nabla u|^\alpha \, dx \\ & - \varepsilon \frac{M_3^{-(\beta-1)}}{\beta} H^{\sigma(\beta-1)}(t) \int_\Omega |\nabla u|^\beta \, dx \\ & - \frac{a\varepsilon}{m} M_1^{-(m-1)} H^{\sigma(m-1)}(t) \int_\Omega |u|^m \, dx + b\varepsilon \int_\Omega |u|^p \, dx \\ & - \varepsilon \left[M_2 \int_\Omega |\nabla u_t|^2 \, dx + \frac{\beta-1}{\beta} M_3 \int_\Omega |\nabla u_t|^\beta \, dx \right. \\ & \left. + a \frac{m-1}{m} M_1 \int_\Omega |u_t|^m \, dx \right] H^{-\sigma}(t) \end{aligned} \tag{15}$$

If $M = M_2 + (\beta - 1) M_3 / \beta + (m-1) M_1 / m$ then (15) takes the form

$$\begin{aligned} L'(t) \geqslant{} & ((1-\sigma) - \varepsilon M) H^{-\sigma}(t) H'(t) + \varepsilon \int_\Omega u_t^2 \, dx \\ & - \frac{\varepsilon}{4M_2} H^\sigma(t) \int_\Omega |\nabla u|^2 \, dx - \varepsilon \int_\Omega |\nabla u|^\alpha \, dx \\ & - \varepsilon \frac{M_3^{-(\beta-1)}}{\beta} H^{\sigma(\beta-1)}(t) \int_\Omega |\nabla u|^\beta \, dx \\ & - \frac{a\varepsilon}{m} M_1^{-(m-1)} H^{\sigma(m-1)}(t) \int_\Omega |u|^m \, dx + b\varepsilon \int_\Omega |u|^p \, dx \end{aligned} \tag{16}$$

We then use the embedding $L^p(\Omega) \hookrightarrow L^m(\Omega)$ and (6) to get

$$H^{\sigma(m-1)}(t) \int_\Omega |u|^m \, dx \leqslant \left(\frac{b}{p} \right)^{\sigma(m-1)} \left(\int_\Omega |u|^p \, dx \right)^{\frac{m+\sigma p(m-1)}{p}} \tag{17}$$

We also exploit the inequality

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x \leqslant C \left(\int_\Omega |\nabla u|^\alpha \, \mathrm{d}x \right)^{2/\alpha}$$

the embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^p(\Omega)$, and (4) to obtain

$$H^\sigma(t) \int_\Omega |\nabla u|^2 \, \mathrm{d}x \leqslant C \left(\frac{b}{p} \right)^\sigma \left(\int_\Omega |\nabla u|^\alpha \, \mathrm{d}x \right)^{\frac{p\sigma+2}{\alpha}} \tag{18}$$

Since $\alpha > \beta$ we have

$$\int_\Omega |\nabla u|^\beta \, \mathrm{d}x \leqslant C \left(\int_\Omega |\nabla u|^\alpha \, \mathrm{d}x \right)^{\beta/\alpha}$$

consequently

$$H^{\sigma(\beta-1)}(t) \int_\Omega |\nabla u|^\beta \, \mathrm{d}x \leqslant C \left(\frac{b}{p} \right)^{\sigma(\beta-1)} \left(\int_\Omega |\nabla u|^\alpha \, \mathrm{d}x \right)^{\frac{p\sigma(\beta-1)+\beta}{\alpha}} \tag{19}$$

where C is a constant depending on Ω only. By using (8) and

$$z^v \leqslant z + 1 \leqslant \left(1 + \frac{1}{a} \right) (z + a), \quad \forall z \geqslant 0,\ 0 < v \leqslant 1,\ a \geqslant 0 \tag{20}$$

we have the following

$$\begin{aligned} \left(\int_\Omega |u|^p \, \mathrm{d}x \right)^{\frac{m+\sigma p(m-1)}{p}} &\leqslant \left(\int_\Omega |\nabla u|^\alpha \, \mathrm{d}x \right)^{\frac{m+\sigma p(m-1)}{\alpha}} \\ &\leqslant d \left(\int_\Omega |\nabla u|^\alpha \, \mathrm{d}x + H(0) \right) \\ &\leqslant d \left(\int_\Omega |\nabla u|^\alpha \, \mathrm{d}x + H(t) \right) \quad \forall t \geqslant 0 \end{aligned} \tag{21}$$

$$\left(\int_\Omega |\nabla u|^\alpha \, \mathrm{d}x \right)^{\frac{p\sigma+2}{\alpha}} \leqslant d \left(\int_\Omega |\nabla u|^\alpha \, \mathrm{d}x + H(t) \right), \quad \forall t \geqslant 0 \tag{22}$$

$$\left(\int_\Omega |\nabla u|^\alpha \, \mathrm{d}x \right)^{\frac{p\sigma(\beta-1)+\beta}{\alpha}} \leqslant d \left(\int_\Omega |\nabla u|^\alpha \, \mathrm{d}x + H(t) \right), \quad \forall t \geqslant 0 \tag{23}$$

where $d = 1 + 1/H(0)$. Inserting the estimates (17)–(19) and (21)–(23) into (16) we get

$$\begin{aligned} L'(t) \geqslant{} & ((1-\sigma) - \varepsilon M)H^{-\sigma}(t)H'(t) \\ & + kH(t) + \left(\varepsilon + \frac{k}{2}\right)\int_\Omega u_t^2\,\mathrm{d}x \\ & - \frac{\varepsilon C_2}{M_2}\left(\int_\Omega |\nabla u|^\alpha\,\mathrm{d}x + H(t)\right) - \varepsilon\int_\Omega |\nabla u|^\alpha\,\mathrm{d}x \\ & - \frac{\varepsilon C_3}{M_3^{\beta-1}}\left(\int_\Omega |\nabla u|^\alpha\,\mathrm{d}x + H(t)\right) + \frac{k}{\alpha}\int_\Omega |\nabla u|^\alpha\,\mathrm{d}x \\ & - \frac{\varepsilon C_1}{M_1^{m-1}}\left(\int_\Omega |\nabla u|^\alpha\,\mathrm{d}x + H(t)\right) + b\left(\varepsilon - \frac{k}{p}\right)\int_\Omega |u|^p\,\mathrm{d}x \end{aligned} \tag{24}$$

for some constant k and

$$C_1 = \frac{aCd}{m}\left(\frac{b}{p}\right)^{\sigma(m-1)}, \quad C_2 = \frac{Cd}{4}\left(\frac{b}{p}\right)^{\sigma}, \quad C_3 = \frac{Cd}{\beta}\left(\frac{b}{p}\right)^{\sigma(\beta-1)}$$

Using $k = \varepsilon p$, we arrive at

$$\begin{aligned} L'(t) \geqslant{} & ((1-\sigma) - \varepsilon M)H^{-\sigma}(t)H'(t) + \varepsilon\left(\frac{p+2}{2}\right)\int_\Omega u_t^2\,\mathrm{d}x \\ & + \varepsilon\left(p - \frac{C_2}{M_2} - \frac{C_3}{M_3^{\beta-1}} - \frac{C_1}{M_1^{m-1}}\right)H(t) \\ & + \varepsilon\left(\frac{p}{\alpha} - \frac{C_2}{M_2} - \frac{C_3}{M_3^{\beta-1}} - \frac{C_1}{M_1^{m-1}} - 1\right)\int_\Omega |\nabla u|^\alpha\,\mathrm{d}x \end{aligned} \tag{25}$$

At this point, we choose M_1, M_2, M_3 large enough so that

$$\begin{aligned} L'(t) \geqslant{} & ((1-\sigma) - \varepsilon M)H^{-\sigma}(t)H'(t) \\ & + \gamma\varepsilon\left[H(t) + \int_\Omega u_t^2\,\mathrm{d}x + \int_\Omega |\nabla u|^\alpha\,\mathrm{d}x\right] \end{aligned} \tag{26}$$

where γ is a positive constant (this is possible since $p > \alpha$). By choosing $\varepsilon < (1-\sigma)/M$ so that

$$L(0) = H^{1-\sigma}(0) + \varepsilon\int_\Omega u_0 u_1\,\mathrm{d}x > 0$$

we obtain

$$L(t) \geqslant L(0) > 0, \quad \forall t \geqslant 0$$

and

$$L'(t) \geqslant \gamma\varepsilon \left[H(t) + \int_\Omega u_t^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^\alpha \,\mathrm{d}x \right] \tag{27}$$

Next, it is clear that

$$L^{\frac{1}{1-\sigma}}(t) \leqslant 2^{\frac{1}{1-\sigma}} \left\{ H(t) + \varepsilon^{\frac{1}{1-\sigma}} \left(\int_\Omega u_t u \,\mathrm{d}x \right)^{\frac{1}{1-\sigma}} \right\}$$

By the Cauchy–Schwarz inequality and the embedding of the $L^p(\Omega)$ spaces we have

$$\begin{aligned} \left| \int_\Omega u_t u \,\mathrm{d}x \right| &\leqslant \left(\int_\Omega u^2 \,\mathrm{d}x \right)^{1/2} \left(\int_\Omega u_t^2 \,\mathrm{d}x \right)^{1/2} \\ &\leqslant C \left(\int_\Omega |u|^\alpha \,\mathrm{d}x \right)^{1/\alpha} \left(\int_\Omega u_t^2 \,\mathrm{d}x \right)^{1/2} \end{aligned}$$

which implies

$$\left| \int_\Omega u_t u \,\mathrm{d}x \right|^{\frac{1}{1-\sigma}} \leqslant C \left(\int_\Omega |u|^\alpha \,\mathrm{d}x \right)^{\frac{1}{(1-\sigma)\alpha}} \left(\int_\Omega u_t^2 \,\mathrm{d}x \right)^{\frac{1}{2(1-\sigma)}}$$

Also Young's inequality gives

$$\left| \int_\Omega u_t u \,\mathrm{d}x \right|^{\frac{1}{1-\sigma}} \leqslant C \left[\left(\int_\Omega |u|^\alpha \,\mathrm{d}x \right)^{\frac{\mu}{(1-\sigma)\alpha}} + \left(\int_\Omega u_t^2 \,\mathrm{d}x \right)^{\frac{\theta}{2(1-\sigma)}} \right]$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\sigma)$, (hence $\mu = 2(1-\sigma)/(1-2\sigma)$) to get

$$\left| \int_\Omega u_t u \,\mathrm{d}x \right|^{\frac{1}{1-\sigma}} \leqslant C \left[\left(\int_\Omega |u|^\alpha \,\mathrm{d}x \right)^{\frac{2}{(1-2\sigma)\alpha}} + \int_\Omega u_t^2 \,\mathrm{d}x \right]$$

By Poincaré's inequality, we obtain

$$\left| \int_\Omega u_t u \,\mathrm{d}x \right|^{\frac{1}{1-\sigma}} \leqslant C \left[\left(\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x \right)^{\frac{2}{(1-2\sigma)\alpha}} + \int_\Omega u_t^2 \,\mathrm{d}x \right]$$

By using (8) and (20) we deduce

$$\left(\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x \right)^{\frac{2}{(1-2\sigma)\alpha}} \leqslant \left(1 + \frac{1}{H(0)} \right) \left(\int_\Omega |\nabla u|^\alpha \,\mathrm{d}x + H(t) \right)$$

Therefore

$$\left|\int_{\Omega} u_t u \, dx\right|^{\frac{1}{1-\sigma}} \leqslant C\left[H(t)+\int_{\Omega}|\nabla u|^{\alpha} \, dx+\int_{\Omega} u_t^2 \, dx\right], \quad \forall t \geqslant 0$$

consequently

$$L^{\frac{1}{1-\sigma}}(t) \leqslant \Gamma\left[H(t)+\int_{\Omega}|\nabla u|^{\alpha} \, dx+\int_{\Omega} u_t^2 \, dx\right] \tag{28}$$

where Γ is positive constant. A combination of (27) and (28), thus, yields

$$L'(t) \geqslant \xi L^{\frac{1}{1-\sigma}}(t), \quad \forall t \geqslant 0 \tag{29}$$

Integration of (29) over $(0, t)$ gives

$$L^{\frac{\sigma}{1-\sigma}}(t) \geqslant \frac{1}{L^{\frac{-\sigma}{1-\sigma}}(0)-\frac{\xi \sigma}{(1-\sigma)} t}$$

hence $L(t)$ blow up in time

$$T^* \leqslant \frac{1-\sigma}{\xi \sigma L^{\frac{\sigma}{1-\sigma}}(0)} \tag{30}$$

Remark 2.3
The time estimate (30) shows that the larger $L(0)$ is the quicker the blow up takes place.

Remark 2.4
In (6) we only require that $H(0)>0$, Unlike Yang [3], where it is required that $H(0)>A$, a constant depending on the size of Ω. See condition (ii), Theorem 2.1 of [3].

Remark 2.5
If we consider

$$\begin{gathered} u_{tt}-\Delta u_t-\operatorname{div}(\sigma(\nabla u) \nabla u)-\operatorname{div}(\beta(\nabla u) \nabla u_t) \\ +f(u_t)=g(u), \quad x \in \Omega, \ t>0 \end{gathered}$$

with the initial and boundary conditions of (1) we can establish a similar blow up result under the growth conditions of Theorem 2.3 of [3] on f, g, σ and β.

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