Global non-existence of solutions of a class of wave equations with non-linear damping and source terms

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SUMMARY

In this paper we consider the non-linear wave equation

\[ u_{tt} - \Delta u_t - \text{div}(|\nabla u|^{\beta-2}\nabla u) - \text{div}(|\nabla u_t|^{\beta-2}\nabla u_t) + a|u_t|^{m-2}u_t = b|u|^{p-2}u \]

\(a, b > 0\), associated with initial and Dirichlet boundary conditions. We prove, under suitable conditions on \(a, \beta, m, p\) and for negative initial energy, a global non-existence theorem. This improves a result by Yang (Math. Meth. Appl. Sci. 2002; 25:825–833), who requires that the initial energy be sufficiently negative and relates the global non-existence of solutions to the size of \(\Omega\). Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: non-linear damping; non-linear source; negative initial energy; global non-existence

1. INTRODUCTION

In this paper we are concerned with the following initial boundary value problem

\[
\begin{aligned}
&u_{tt} - \Delta u_t - \text{div}(|\nabla u|^{\beta-2}\nabla u) - \text{div}(|\nabla u_t|^{\beta-2}\nabla u_t) \\
&+ a|u_t|^{m-2}u_t = b|u|^{p-2}u, \quad x \in \Omega, \ t > 0 \\
&u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \ x \in \Omega \\
&u(x, t) = 0, \quad x \in \partial\Omega, \ t > 0
\end{aligned}
\]

(1)

where \(a, b > 0\), \(\alpha, \beta, m, p > 2\), and \(\Omega\) is a bounded domain of \(\mathbb{R}^n (n \geq 1)\), with a smooth boundary \(\partial\Omega\).

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Equation (1) appears in the models of non-linear viscoelasticity (see References [1–3]). It also can be considered as a system governing the longitudinal motion of a viscoelastic configuration obeying a non-linear Voight model (see References [3,4]).

In the absence of viscosity and strong damping, Equation (1) becomes

$$u_t - \text{div}(|\nabla u|^{p-2}\nabla u) + a|u|^{m-2}u_t = b|u|^{p-2}u, \quad x \in \Omega, \; t > 0$$

(2)

For $b = 0$, it is well known that the damping term assures global existence and decay of the solution energy for arbitrary initial data (see References [5,6]). Then, for $a = 0$ the source term causes finite time blow up of solutions with negative initial energy if $p > \alpha$ (see References [7,8]).

The interaction between the damping and the source terms was first considered by Levine [9,10] in the linear damping case ($\alpha = m = 2$). He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [11] extended Levine’s result to the non-linear damping case ($m > 2$). In their work, the authors considered (2) with $\alpha = 2$ and introduced a method different than the one known as the concavity method. They determined suitable relations between $m$ and $p$, for which there is global existence or alternatively finite time blow up. Precisely; they showed that solutions with negative energy continue to exist globally ‘in time’ if $m > p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative. This result was later generalized to an abstract setting and to unbounded domains by Levine and Serrin [12] and Levine et al. [13]. In these papers, the authors showed that no solution with negative energy can be extended on $[0, \infty)$ if $p > m$ and proved several non-continuation theorems. This generalization allowed them also to apply their result to quasilinear situations ($\alpha > 2$), of which the problem in Reference [11] is a particular case. Vitillaro [14] combined the arguments in References [11,12] to extend these results to situations where the damping is non-linear and the solution has positive initial energy. Similar results have also been established by Todorova [15,16] for different Cauchy problems.

In Reference [3], Yang studied (1) and proved a blow up result under the condition $p > \max\{\alpha, m\}$, $\alpha > \beta$, and the initial energy is sufficiently negative (see condition (ii) Theorem 2.1 of Reference [3]). In fact this condition made it clear that there exists a certain relation between the blow-up time and $|\Omega|$ (see Remark 2 of Reference [3]). We should note here that (1) corresponds to Equation (5) of [3] but the same conclusions hold for Equation (1) of the same paper, under suitable conditions, stated in Theorem 2.3 of [3].

In this work we show that any weak solution of (1), with negative initial energy, cannot exists for all time if $p > \max\{\alpha, m\}$, $\alpha > \beta$. Therefore, our result improves the one of [3]. Our technique of proof follows closely the argument of [17] with the modifications needed for our problem.

2. BLOW UP

In order to state and prove our result, we introduce the following function space

$$Z = L^\infty([0, T); W^{1,2}_0(\Omega)) \cap W^{1,\infty}([0, T); L^2(\Omega))$$

$$\cap W^{1,\beta}([0, T); W^{1,\beta}_0(\Omega)) \cap W^{1,m}([0, T); L^m(\Omega))$$
for $T > 0$ and the energy functional

$$E(t) = \frac{1}{2} \int_{\Omega} u_t^2 \, dx + \frac{1}{\alpha} \int_{\Omega} |\nabla u|^2 \, dx - \frac{b}{p} \int_{\Omega} |u|^p \, dx$$  \hspace{1cm} (3)

**Theorem**

Assume that $\alpha, \beta, m, p \geq 2$ such that $\beta < \alpha$, and $\max\{m, \alpha\} < p < r_\alpha$, where $r_\alpha$ is the Sobolev critical exponent of $W^{1,2}_0(\Omega)$. Assume further that

$$E(0) < 0$$  \hspace{1cm} (4)

Then the solution $u \in Z$, of (1), cannot exist for all time.

**Remark 2.1**

We remind that $r_\alpha = \frac{n}{\alpha - 1}$, if $n > \alpha$, $r_\alpha > \alpha$ if $n = \alpha$, and $r_\alpha = \infty$ if $n < \alpha$.

**Remark 2.2**

If the solution $u$ is smooth enough then it blows up in finite time.

**Proof**

We suppose that the solution exists for all time and we reach to a contradiction. For this purpose we multiply Equation (1) by $u_t$ and integrate over $\Omega$ to obtain

$$E'(t) = -\int_{\Omega} |\nabla u_t|^2 \, dx - \int_{\Omega} |\nabla u|^\beta \, dx - a \int_{\Omega} |u|^m \, dx \leq 0$$  \hspace{1cm} (5)

for any regular solution. This remains valid for $u \in Z$ by density argument. Hence $E(t) \leq E(0)$, $\forall t \geq 0$.

By setting $H(t) = -E(t)$, we get

$$0 < H(0) \leq H(t) \leq \frac{b}{p} \int_{\Omega} |u|^p \, dx, \hspace{1cm} \forall t \geq 0$$  \hspace{1cm} (6)

We then define

$$L(t) = H^{1-\sigma}(t) + \varepsilon \int_{\Omega} uu_t \, dx$$  \hspace{1cm} (7)

for $\varepsilon$ small to be chosen later and

$$0 < \sigma \leq \min\left(\frac{\alpha - 2}{p}, \frac{\alpha - \beta}{p(\beta - 1)}, \frac{p - m}{p(m - 1)}, \frac{\alpha - 2}{2 \alpha}\right)$$  \hspace{1cm} (8)

Our goal is to show that $L(t)$ satisfies a differential inequality of the form

$$L'(t) \geq \xi L^q(t), \hspace{1cm} q > 1$$

This, of course, will lead to a blow up in finite time.

By taking a derivative of (7) we obtain

$$L'(t) = (1 - \sigma)H^{-\sigma}(t)H'(t) + \varepsilon \int_{\Omega} u_t^2 \, dx + \varepsilon \int_{\Omega} uu_{tt} \, dx$$  \hspace{1cm} (9)
By using Equation (1), the estimate (9) gives

\[ L'(t) = (1 - \sigma)H^{-\sigma}(t)H'(t) + \varepsilon \int_\Omega u_t^2 \, dx \]

\[ - \varepsilon \int_\Omega \nabla u \nabla u_t \, dx - \varepsilon \int_\Omega |\nabla u|^2 \, dx \]

\[ - \varepsilon \int_\Omega |\nabla u_t|^{\beta-2} \nabla u_t \nabla u \, dx \]

\[ - a \varepsilon \int_\Omega |u_t|^{m-2} u_t u \, dx + b \varepsilon \int_\Omega |u|^p \, dx \]

\[ (10) \]

We then exploit Young’s inequality to get

\[ \int_\Omega |u_t|^{m-2} u_t u \, dx \leq \frac{\delta^m}{m} \int_\Omega |u|^m \, dx + \frac{m-1}{m} \delta^{-m(m-1)} \int_\Omega |u_t|^m \, dx \]

\[ (11) \]

\[ \int_\Omega \nabla u \nabla u_t \, dx \leq \frac{1}{4\mu} \int_\Omega |\nabla u|^2 \, dx + \mu \int_\Omega |\nabla u_t|^2 \, dx \]

\[ (12) \]

\[ \int_\Omega |\nabla u_t|^{\beta-1} \nabla u \, dx \leq \frac{\lambda^\beta}{\beta - 1} \int_\Omega |\nabla u|^\beta \, dx + \frac{\beta - 1}{\beta} \lambda^{-\beta/(\beta-1)} \int_\Omega |\nabla u_t|^\beta \, dx \]

\[ (13) \]

A substitution of (11)–(13) in (10) yields

\[ L'(t) \geq (1 - \sigma)H^{-\sigma}(t)H'(t) + \varepsilon \int_\Omega u_t^2 \, dx \]

\[ - \frac{\varepsilon}{4\mu} \int_\Omega |\nabla u|^2 \, dx - \mu \varepsilon \int_\Omega |\nabla u_t|^2 \, dx \]

\[ - \varepsilon \int_\Omega |\nabla u|^2 \, dx - \varepsilon \frac{\lambda^\beta}{\beta} \int_\Omega |\nabla u|^\beta \, dx \]

\[ - \varepsilon \frac{\beta - 1}{\beta} \lambda^{-\beta/(\beta-1)} \int_\Omega |\nabla u_t|^\beta \, dx \]

\[ + b \varepsilon \int_\Omega |u|^p \, dx - a \varepsilon \frac{\delta^m}{m} \int_\Omega |u|^m \, dx \]

\[ - a \varepsilon \frac{m-1}{m} \delta^{-m(m-1)} \int_\Omega |u_t|^m \, dx \]

\[ (14) \]
Therefore by choosing $\delta, \mu, \lambda$ so that

$$
\begin{align*}
\delta^{-m/(m-1)} &= M_1 H^{-\sigma}(t) \\
\mu &= M_2 H^{-\sigma}(t) \\
\lambda^{-\beta/(\beta-1)} &= M_3 H^{-\sigma}(t)
\end{align*}
$$

for $M_1, M_2,$ and $M_3$ to be specified later, and using (14) we arrive at

$$
L'(t) \geq (1 - \sigma) H^{-\sigma}(t) H'(t) + \epsilon \int_{\Omega} u_t^2 \, dx \\
- \frac{\epsilon}{4M_2} H^\sigma(t) \int_{\Omega} |\nabla u|^2 \, dx - \epsilon \int_{\Omega} |\nabla u|^\sigma \, dx \\
- \frac{\epsilon}{\beta} M_3^{-\frac{(\beta-1)}{\beta}} H^{\sigma(\beta-1)}(t) \int_{\Omega} |\nabla u|^{\beta} \, dx \\
- \frac{ae}{m} M_1^{-\frac{(m-1)}{m}} H^{\sigma(m-1)}(t) \int_{\Omega} |u|^m \, dx + b \epsilon \int_{\Omega} |u|^p \, dx \\
- \epsilon \left[ M_2 \int_{\Omega} |\nabla u_t|^2 \, dx + \frac{\beta - 1}{\beta} M_3 \int_{\Omega} |\nabla u_t|^{\beta} \, dx \right] H^{-\sigma}(t)
\right]
$$

(15)

If $M = M_2 + (\beta - 1)M_3/\beta + (m-1)M_1/m$ then (15) takes the form

$$
L'(t) \geq \left( (1 - \sigma) - \epsilon M \right) H^{-\sigma}(t) H'(t) + \epsilon \int_{\Omega} u_t^2 \, dx \\
- \frac{\epsilon}{4M_2} H^\sigma(t) \int_{\Omega} |\nabla u|^2 \, dx - \epsilon \int_{\Omega} |\nabla u|^\sigma \, dx \\
- \frac{\epsilon}{\beta} M_3^{-\frac{(\beta-1)}{\beta}} H^{\sigma(\beta-1)}(t) \int_{\Omega} |\nabla u|^{\beta} \, dx \\
- \frac{ae}{m} M_1^{-\frac{(m-1)}{m}} H^{\sigma(m-1)}(t) \int_{\Omega} |u|^m \, dx + b \epsilon \int_{\Omega} |u|^p \, dx
\right]
$$

(16)

We then use the embedding $L^p(\Omega) \hookrightarrow L^m(\Omega)$ and (6) to get

$$
H^{\sigma(m-1)}(t) \int_{\Omega} |u|^m \, dx \leq \left( \frac{b}{p} \right) ^{\sigma(m-1)} \left( \int_{\Omega} |u|^p \, dx \right) ^{\frac{m + \sigma p(m-1)}{p}}
$$

(17)
We also exploit the inequality
\[ \int_{\Omega} |\nabla u|^2 \, dx \leq C \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{2/\alpha} \]
the embedding \( W_0^{1,\alpha}(\Omega) \rightarrow L^\beta(\Omega) \), and (4) to obtain
\[ H^\alpha(t) \int_{\Omega} |\nabla u|^2 \, dx \leq C \left( \frac{b}{p} \right)^\sigma \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{ps+2}{2}} \] (18)
Since \( \alpha > \beta \) we have
\[ \int_{\Omega} |\nabla u|^\beta \, dx \leq C \left( \int_{\Omega} |\nabla u|^\alpha \, dx \right)^{\frac{\beta}{\alpha}} \]
consequently
\[ H^{\alpha(\beta-1)}(t) \int_{\Omega} |\nabla u|^\beta \, dx \leq C \left( \frac{b}{p} \right)^{\sigma(\beta-1)} \left( \int_{\Omega} |\nabla u|^\alpha \, dx \right)^{\frac{ps(\beta-1)+\beta}{2}} \] (19)
where \( C \) is a constant depending on \( \Omega \) only. By using (8) and
\[ z^v \leq z + 1 \leq \left( 1 + \frac{1}{a} \right) (z + a), \; \forall z \geq 0, \; 0 < v \leq 1, \; a \geq 0 \] (20)
we have the following
\[
\left( \int_{\Omega} |u|^p \, dx \right)^{\frac{m+\sigma p(m-1)}{p}} \leq \left( \int_{\Omega} |\nabla u|^\alpha \, dx \right)^{\frac{m+\sigma p(m-1)}{\alpha}} \\
\leq d \left( \int_{\Omega} |\nabla u|^\alpha \, dx + H(0) \right) \\
\leq d \left( \int_{\Omega} |\nabla u|^\alpha \, dx + H(t) \right), \; \forall t \geq 0 \] (21)
\[
\left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{ps+2}{2}} \leq d \left( \int_{\Omega} |\nabla u|^\alpha \, dx + H(t) \right), \; \forall t \geq 0 \] (22)
\[
\left( \int_{\Omega} |\nabla u|^\beta \, dx \right)^{\frac{ps(\beta-1)+\beta}{\alpha}} \leq d \left( \int_{\Omega} |\nabla u|^\alpha \, dx + H(t) \right), \; \forall t \geq 0 \] (23)
where \( d = 1 + 1/H(0) \). Inserting the estimates (17)–(19) and (21)–(23) into (16) we get

\[
L'(t) \geq ((1 - \sigma) - \varepsilon M)H^{-\sigma}(t)H'(t) + kH(t) + \left( \varepsilon \frac{k}{2} \right) \int_{\Omega} u_t^2 \, dx
\]

\[
- \frac{\varepsilon C_2}{M_2} \left( \int_{\Omega} |\nabla u|^2 \, dx + H(t) \right) - \varepsilon \int_{\Omega} |\nabla u|^2 \, dx
\]

\[
- \frac{\varepsilon C_3}{M_3^{\beta-1}} \left( \int_{\Omega} |\nabla u|^2 \, dx + H(t) \right) + \frac{k}{\alpha} \int_{\Omega} |\nabla u|^2 \, dx
\]

\[
- \frac{\varepsilon C_1}{M_1^{\beta^{-1}}} \left( \int_{\Omega} |\nabla u|^2 \, dx + H(t) \right) + b \left( \varepsilon - \frac{k}{p} \right) \int_{\Omega} |u|^p \, dx
\]

(24)

for some constant \( k \) and

\[
C_1 = \frac{aCd}{m} \left( \frac{b}{p} \right)^{\sigma(m-1)}, \quad C_2 = \frac{Cd}{4} \left( \frac{b}{p} \right)^{\sigma}, \quad C_3 = \frac{Cd}{\beta} \left( \frac{b}{p} \right)^{\sigma(\beta-1)}
\]

Using \( k = \varepsilon p \), we arrive at

\[
L'(t) \geq ((1 - \sigma) - \varepsilon M)H^{-\sigma}(t)H'(t) + \varepsilon \left( \frac{p + 2}{2} \right) \int_{\Omega} u_t^2 \, dx
\]

\[
+ \varepsilon \left( p - \frac{C_2}{M_2} - \frac{C_3}{M_3^{\beta-1}} - \frac{C_1}{M_1^{\beta^{-1}}} \right) H(t)
\]

\[
+ \varepsilon \left( \frac{p}{\alpha} - \frac{C_2}{M_2} - \frac{C_3}{M_3^{\beta-1}} - \frac{C_1}{M_1^{\beta^{-1}}} - 1 \right) \int_{\Omega} |\nabla u|^2 \, dx
\]

(25)

At this point, we choose \( M_1, M_2, M_3 \) large enough so that

\[
L'(t) \geq ((1 - \sigma) - \varepsilon M)H^{-\sigma}(t)H'(t)
\]

\[
+ \frac{\gamma \varepsilon}{\delta} H(t) + \int_{\Omega} u_t^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx
\]

(26)

where \( \gamma \) is a positive constant (this is possible since \( p > \alpha \)). By choosing \( \varepsilon < (1 - \sigma)/M \) so that

\[
L(0) = H^{1-\sigma}(0) + \varepsilon \int_{\Omega} u_0 u_1 \, dx > 0
\]

we obtain

\[
L(t) \geq L(0) > 0, \quad \forall t \geq 0
\]
and

\[ L'(t) \geq \gamma e \left[ H(t) + \int_{\Omega} u_t^2 \, dx + \int_{\Omega} |\nabla u|^\gamma \, dx \right] \]  \tag{27}

Next, it is clear that

\[ L^{1-\sigma}(t) \leq 2^{1-\sigma} \left\{ H(t) + \varepsilon^{1-\sigma} \left( \int_{\Omega} u_t u \, dx \right)^{1-\sigma} \right\} \]

By the Cauchy–Schwarz inequality and the embedding of the \( L^p(\Omega) \) spaces we have

\[ \left| \int_{\Omega} u_t u \, dx \right| \leq \left( \int_{\Omega} u^2 \, dx \right)^{1/2} \left( \int_{\Omega} u_t^2 \, dx \right)^{1/2} \leq C \left( \int_{\Omega} |u|^\gamma \, dx \right)^{1/\gamma} \left( \int_{\Omega} u_t^2 \, dx \right)^{1/2} \]

which implies

\[ \left| \int_{\Omega} u_t u \, dx \right|^{1-\sigma} \leq C \left( \int_{\Omega} |u|^\gamma \, dx \right)^{1/(1-\sigma)\gamma} \left( \int_{\Omega} u_t^2 \, dx \right)^{\frac{1}{2(1-\sigma)}} \]

Also Young’s inequality gives

\[ \left| \int_{\Omega} u_t u \, dx \right|^{1-\sigma} \leq C \left[ \left( \int_{\Omega} |u|^\gamma \, dx \right)^{\frac{\mu}{(1-\sigma)\gamma}} + \left( \int_{\Omega} u_t^2 \, dx \right)^{\frac{\theta}{2(1-\sigma)}} \right] \]

for \( 1/\mu + 1/\theta = 1 \). We take \( \theta = 2(1 - \sigma) \), (hence \( \mu = 2(1 - \sigma)/(1 - 2\sigma) \)) to get

\[ \left| \int_{\Omega} u_t u \, dx \right|^{1-\sigma} \leq C \left[ \left( \int_{\Omega} |u|^\gamma \, dx \right)^{\frac{2}{(1-2\sigma)\gamma}} + \int_{\Omega} u_t^2 \, dx \right] \]

By Poincaré’s inequality, we obtain

\[ \left| \int_{\Omega} u_t u \, dx \right|^{1-\sigma} \leq C \left[ \left( \int_{\Omega} |\nabla u|^\gamma \, dx \right)^{\frac{2}{(1-2\sigma)\gamma}} + \int_{\Omega} u_t^2 \, dx \right] \]

By using (8) and (20) we deduce

\[ \left( \int_{\Omega} |\nabla u|^\gamma \, dx \right)^{\frac{2}{(1-2\sigma)\gamma}} \leq \left( 1 + \frac{1}{H(0)} \right) \left( \int_{\Omega} |\nabla u|^\gamma \, dx + H(t) \right) \]
Therefore
\[ \left| \int_{\Omega} u_t u \, dx \right|^{\frac{1}{1-\sigma}} \leq C \left[ H(t) + \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} u_t^2 \, dx \right], \quad \forall t \geq 0 \]
consequently
\[ L^{\frac{1}{1-\sigma}}(t) \leq \Gamma \left[ H(t) + \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} u_t^2 \, dx \right] \tag{28} \]
where \( \Gamma \) is positive constant. A combination of (27) and (28), thus, yields
\[ L'(t) \geq \xi L^{\frac{1}{1-\sigma}}(t), \quad \forall t \geq 0 \tag{29} \]
Integration of (29) over \((0, t)\) gives
\[ L^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{L^{\frac{\sigma}{1-\sigma}}(0) - \frac{\xi \sigma}{(1-\sigma)t}} \]
hence \( L(t) \) blow up in time
\[ T^* \leq \frac{1 - \sigma}{\xi \sigma L^{\frac{\sigma}{1-\sigma}}(0)} \tag{30} \]

**Remark 2.3**
The time estimate (30) shows that the larger \( L(0) \) is the quicker the blow up takes place.

**Remark 2.4**
In (6) we only require that \( H(0) > 0 \), Unlike Yang [3], where it is required that \( H(0) > A \), a constant depending on the size of \( \Omega \). See condition (ii), Theorem 2.1 of [3].

**Remark 2.5**
If we consider
\[ u_{tt} - \Delta u_t - \nabla(\sigma(\nabla u)\nabla u) - \nabla(\beta(\nabla u)\nabla u_t) + f(u_t) = g(u), \quad x \in \Omega, \ t > 0 \]
with the initial and boundary conditions of (1) we can establish a similar blow up result under the growth conditions of Theorem 2.3 of [3] on \( f, g, \sigma \) and \( \beta \).

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**REFERENCES**
1. Andrews G. On the existence of solutions to the equation \( u_{tt} - u_{xx} = \sigma(u_x)_x \). *Journal of Differential Equations* 1980; 35:200–231.