Global Nonexistence in a Nonlinearly Damped Wave Equation

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In this paper, we consider the nonlinearly damped semilinear wave equation

\[ u_{tt} - \Delta u + au_t (1 + |u_t|^{m-2}) = bu|u|^{p-2} \]

associated with initial and Dirichlet boundary conditions. We prove that any solution, with sufficiently negative initial energy, blows up in finite time if \( p > m \).

Keywords: Nonlinear damping; Sufficiently negative; Initial energy; Noncontinuation; Blow up; Finite time

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1 INTRODUCTION

In this paper, we are concerned with the following initial boundary value problem

\[ u_{tt} - \Delta u + au_t (1 + |u_t|^{m-2}) = bu|u|^{p-2}, \quad x \in \Omega, \quad t > 0 \]

\[ u(x, t) = 0, \quad x \in \partial \Omega, \quad t \geq 0 \]

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (1) \]

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where $a, b > 0$, $p, m > 2$, and $\Omega$ is bounded domain of $\mathbb{R}^n (n \geq 1)$, with a smooth boundary $\partial \Omega$. For $b = 0$, it is well known that the damping term assures global existence for arbitrary initial data (see [3,5]). If $a = 0$, then the source term $bu|u|^{p-2}$ causes finite time blow up of solutions with negative initial energy (see [1,4,6,7]).

The interaction between the damping and the source terms was first considered by Levine [6,7] in linear damping case ($m = 2$). He showed that solutions with negative initial energy blow up in finite time. Recently Georgiev and Todorova [2] extended Levine’s result to the nonlinear case, where the damping term is given by $au|u|^{m-2}$, $m > 2$. In their work, the authors introduced a different method than the famous concavity method and determined suitable relations between $m$ and $p$, for which there is global existence or alternatively finite time blow up. Precisely, they showed that solutions with negative energy continue to exist globally ‘in time’ if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative.

This result has been lately generalized to an abstract setting and to unbounded domains by Levine and Serrin [8] and Levine et al. [9]. In these papers, the authors showed that no solution with negative initial energy can be extended on $[0, \infty)$ if $p > m$ and proved several noncontinuation theorems. This generalization allowed them also to apply their noncontinuation results to quasilinear situations, of which the result of [2] is a particular case. As they pointed out in [8], their result improved the result of [2] by not requiring that the initial energy be sufficiently negative but on the other hand the result of [2] showed that the noncontinuation implies a blow up. Messaoudi [11] used a slightly different technique to prove the same result of [2] without imposing the condition that the initial energy is sufficiently negative.

Vitillaro [12] combined the arguments in [2] and [8] to extend these results to situations where the damping is nonlinear and the solution has positive initial energy.

In this work we show that any solution of (1) with sufficiently negative initial energy blows up in finite time. We shall start by stating a local result, which is considered to be standard for such problems (see [10]).

\textbf{Theorem 1.1} \quad \text{Suppose that } m \geq 2, \ p > 2, \text{ and}

\[ p \leq 2 \frac{n - 1}{n - 2}, \quad n \geq 3. \]
Assume further that
\[(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\]  
(3)

Then the problem (1) has a unique local solution
\[u \in C((0, T); H^1_0(\Omega)), \quad u_t \in C([0, T); L^2(\Omega)) \cap L^m(\Omega \times (0, T)),\]  
(4)

T is small.

2 MAIN RESULT

In this section, we show that the solution (4) blows up in finite time if \( p > m \) and \(-E(0)\) is sufficiently large, where
\[E(t) := \frac{1}{2} \int_{\Omega} [u_t^2 + |\nabla u|^2](x, t)dx - \frac{b}{p} \int_{\Omega} |u(x, t)|^p dx.\]  
(5)

**Lemma 2.1** Suppose that (2) holds. Then there exists a positive constant \( C > 1 \) depending on \( \Omega \) only such that
\[\|u\|_p^s \leq C(\|\nabla u\|_2^2 + \|u\|_p^p)\]  
(6)

for any \( u \in H^1_0(\Omega) \) and \( 2 \leq s \leq p \).

**Proof** If \( \|u\|_p \leq 1 \) then \( \|u\|_p^s \leq \|u\|_p^2 \leq C\|\nabla u\|_2^2 \) by Sobolev embedding theorems. If \( \|u\|_p > 1 \) then \( \|u\|_p^s \leq \|u\|_p^p \). Therefore (6) follows.

We set
\[H(t) := -E(t)\]

and use, throughout this paper, \( C \) to denote a generic positive constant depending on \( a, m, p, \) and \( \Omega \) only. As a result of (5), (6), and the lemma, we have

**Corollary 2.2** Let the assumptions of the lemma hold. Then we have
\[\|u\|_p^s \leq C\left( |H(t)| + \|u_t\|_2^2 + \|u\|_p^p \right)\]  
(7)

for any \( u \in H^1_0(\Omega) \) and \( 2 \leq s \leq p \).
**Theorem 2.3** Let the conditions of the Theorem 1.1 be fulfilled. Assume further that \( p > m \) and \(-E(0)\) is large enough then the solution (4) blows up in finite time.

**Proof**
We multiply Eq. (1.1) by \( u_t \) and integrate over \( \Omega \) to get

\[
E'(t) = -a \left[ \int_{\Omega} |u_t(x, t)|^m dx + \int_{\Omega} |u_t(x, t)|^2 dx \right],
\]

(8)

for any regular solution of (1). This identity remains valid for solutions (4) by a simple density argument. So we have

\[
0 < H(0) \leq H(t) \leq \frac{b}{p} \| u \|_p^p.
\]

(9)

We then define

\[
L(t) := e^{at} H^{1-a}(t) + \epsilon e^{at} \int_{\Omega} uu_t(x, t) dx
\]

(10)

for \( \epsilon \) small to be chosen late and

\[
0 < \alpha \leq \min \left\{ \frac{(p - 2)}{2p}, \frac{(p - m)}{p(m - 1)} \right\}.
\]

(11)

By taking a derivative of (10) and using Eq. (1) we obtain

\[
L'(t) := e^{at} (1 - \alpha) H^{-\alpha}(t) H'(t) + ae^{at} H^{1-\alpha}(t) + \epsilon e^{at} \int_{\Omega} [u_t^2 - |\nabla u|^2](x, t) dx
\]

\[
+ \epsilon \epsilon e^{at} \int_{\Omega} |u(x, t)|^p dx - a \epsilon \int_{\Omega} |u_t|^{m-2} |u_t u(x, t)| dx.
\]

(12)

We then exploit Young’s inequality

\[
XY \leq \frac{\delta r}{r} X^r + \frac{\delta - q}{q} Y^q, \quad X, Y, \geq 0, \quad \delta > 0, \quad \frac{1}{r} + \frac{1}{q} = 1
\]
with \( r = m \) and \( q = m/(m - 1) \) to estimate the last term in (12) as follows

\[
\int_{\Omega} |u_i|^{m-1}|u|dx \leq \frac{\delta^m}{m} \|u\|^m_m + \frac{m-1}{m} \delta^{-m/(m-1)} \|u_t\|^m_m
\]

which yields, by substitution in (12),

\[
L'(t) \geq e^{at} \left[ (1 - \alpha)H^{-\alpha}(t)H'(t) - a \frac{m-1}{m} \epsilon \delta^{-m/(m-1)} \|u_t\|^m_m \right] + \epsilon e^{at} \left[ pH(t) + \frac{p}{2} \int_{\Omega} [u_i^2 + |\nabla u|^2](x, t)dx \right] + \epsilon e^{at} \int_{\Omega} [u_i^2 - |\nabla u|^2](x, t)dx - \epsilon ae^{at} \frac{\delta^m}{m} \|u\|^m_m, \quad \forall \delta > 0. \tag{13}
\]

By noting that \( H'(t) \geq a \|u\|^m_m \), we get

\[
L'(t) \geq e^{at} \left[ (1 - \alpha)H^{-\alpha}(t) - a \frac{m-1}{m} \epsilon \delta^{-m/(m-1)} \right] H'(t) + \epsilon e^{at} \left[ \frac{p}{2} - 1 \right] \int_{\Omega} |\nabla u|^2(x, t)dx + \epsilon e^{at} pH(t) + \epsilon e^{at} \left[ \frac{p}{2} + 1 \right] \int_{\Omega} u_i^2(x, t)dx - \epsilon ae^{at} \frac{\delta^m}{m} \|u\|^m_m, \quad \forall \delta > 0. \tag{14}
\]

We then take \( \delta \) so that \( \delta^{-m/(m-1)} = kH^{-\alpha}(t) \), for large \( k \) to be specified later, and substitute in (14) to arrive at

\[
L'(t) \geq e^{at} \left[ (1 - \alpha) - a \frac{m-1}{m} \epsilon k \right] H^{-\alpha}(t)H'(t) + \epsilon \left( \frac{p}{2} + 1 \right) e^{at} \int_{\Omega} u_i^2(x, t)dx + \epsilon \left( \frac{p}{2} - 1 \right) e^{at} \int_{\Omega} |\nabla u|^2(x, t)dx + \epsilon e^{at} \left[ pH(t) - \frac{k^{1-m}}{m} aH^\alpha(m-1)(t) \|u\|^m_m \right]. \tag{15}
\]

By exploiting (9) and the inequality \( \|u\|^m_m \leq C \|u\|^m_p \), we obtain

\[
H^\alpha(m-1)(t) \|u\|^m_m \leq \left( \frac{b}{\epsilon} \right)^{\alpha(m-1)} C \|u\|^m_p a^\alpha(p(m-1)),
\]
hence (15) yields

\[ L'(t) \geq e^{at} \left[ (1 - \alpha) - \frac{m-1}{m} \varepsilon k \right] H^{-\alpha}(t) H'(t) + e^{at} \int_{\Omega} u_t^2(x, t) \, dx \]

\[ + \varepsilon \left( \frac{p}{2} - 1 \right) e^{at} \int_{\Omega} |\nabla u|^2(x, t) \, dx \]

\[ + \varepsilon e^{at} \left[ pH(t) - \frac{k^{1-m}}{m} \alpha \left( \frac{b}{p} \right)^{\alpha(m-1)} C \|u\|_p^{m+\alpha p(m-1)} \right]. \quad (16) \]

We then use Corollary 2.2 and (11), for \( s = m + \alpha p(m-1) < p \), to deduce from (16)

\[ L'(t) \geq e^{at} \left[ (1 - \alpha) - \frac{m-1}{m} \varepsilon k \right] H^{-\alpha}(t) H'(t) \]

\[ + \varepsilon \left( \frac{p}{2} + 1 \right) e^{at} \int_{\Omega} u_t^2(x, t) \, dx + \varepsilon \left( \frac{p}{2} - 1 \right) e^{at} \int_{\Omega} |\nabla u|^2(x, t) \, dx \]

\[ + \varepsilon e^{at} \left[ pH(t) - \frac{k^{1-m}}{m} \alpha \left( \frac{b}{p} \right)^{\alpha(m-1)} C \{ H(t) + \|u_t\|_2^2 + \|u\|_p^p \} \right]. \]

(17)

At this point, we choose \( k \) large enough so that (17) becomes

\[ L'(t) \geq e^{at} \left[ (1 - \alpha) - \frac{m-1}{m} \varepsilon k \right] H^{-\alpha}(t) H'(t) + Ce^{at} \left[ H(t) + \|u_t\|_2^2 + \|u\|_p^p \right]. \quad (18) \]

We then pick \( \varepsilon > 0 \) small enough so that \( (1 - \alpha) - \varepsilon k(m-1)/m \geq 0 \) and \( L(0) > 0 \). Therefore (18) takes the form

\[ L'(t) \geq Ce^{at} \left[ H(t) + \|u_t\|_2^2 + \|u\|_p^p \right]; \quad (19) \]

hence

\[ L(t) \geq L(0) > 0, \quad \forall t \geq 0. \quad (20) \]
Next we estimate the second term in (10) as follows

$$\left| \int_{\Omega} uu_t(x,t) dx \right| \leq \|u\|_2 \|u_t\|_2 \leq C \|u\|_p \|u_t\|_2$$

which implies

$$\left| \int_{\Omega} uu_t(x,t) dx \right|^{1/(1-\alpha)} \leq C \|u\|_p^{1/(1-\alpha)} \|u_t\|_2^{1/(1-\alpha)}.$$ 

Again Young's inequality gives

$$\left| \int_{\Omega} uu_t(x,t) dx \right|^{1/(1-\alpha)} \leq C \left[ \|u\|_p^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)} \right], \tag{21}$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\alpha)$, to get $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p$ by (11). Therefore (21) becomes

$$\left| \int_{\Omega} uu_t(x,t) dx \right|^{1/(1-\alpha)} \leq C \left[ \|u\|_p^{s} + \|u_t\|_2^{2} \right],$$

where $s = 2/(1-2\alpha) \leq p$. By using Corollary 2.2 we obtain

$$\left| \int_{\Omega} uu_t(x,t) dx \right|^{1/(1-\alpha)} \leq \Gamma \left[ H(t) + \|u\|_p^{p} + \|u_t\|_2^{2} \right], \quad \forall t \geq 0. \tag{22}$$

Consequently we have

$$L^{1/(1-\alpha)}(t) = \left( e^{at} H^{1-\alpha}(t) + \epsilon e^{at} \int_{\Omega} uu_t(x,t) dx \right)^{1/(1-\alpha)}$$

$$\leq 2^{1/(1-\alpha)} e^{at/(1-\alpha)} \left( H(t) + \left| \int_{\Omega} uu_t(x,t) dx \right|^{1/(1-\alpha)} \right), \tag{23}$$

$$\leq C e^{at/(1-\alpha)} \left( H(t) + \|u\|_p^{p} + \|u_t\|_2^{2} \right).$$

We then combine (19) and (23), to arrive at

$$L'(t) \geq C e^{-at/(1-\alpha)} L^{1/(1-\alpha)}(t) \tag{24}$$
A simple integration of (24) over \((0, t)\) yields

\[
L^{-\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - C(1 - e^{-\alpha t/(1-\alpha)})}
\]  

(25)

Therefore by choosing the initial data so that \(L^{\alpha/(1-\alpha)}(0) > 1/C\), \(L(t)\) blows up in a time

\[
T^* \leq \frac{(1 - \alpha) \ln[C/(C - L^{-\alpha/(1-\alpha)}(0))]}{\alpha a}
\]

(26)

**Remark 2.1** The estimate (26) shows that the larger \(L(0)\) is, the quicker the blow up takes place.

**Remark 2.2** By following the steps of the proof of Theorem 2.3 closely, one can easily see that this result holds for \(1 < m < p, p > 2\). Therefore this method is a unified one to both linear and nonlinear damping cases.

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**References**


