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Global Nonexistence in a Nonlinearly Damped Wave Equation

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In this paper, we consider the nonlinearly damped semilinear wave equation

$$u_{tt} - \Delta u + au_t(1 + |u_t|^{m-2}) = bu|u|^{p-2}$$

associated with initial and Dirichlet boundary conditions. We prove that any solution, with sufficiently negative initial energy, blows up in finite time if $p > m$.

Keywords: Nonlinear damping; Sufficiently negative; Initial energy; Noncontinuation; Blow up; Finite time

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1 INTRODUCTION

In this paper, we are concerned with the following initial boundary value problem

$$\begin{aligned} &u_{tt} - \Delta u + au_t(1 + |u_t|^{m-2}) = bu|u|^{p-2}, \quad x \in \Omega, \quad t > 0 \\ &u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0 \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{aligned} \tag{1}$$

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where $a, b > 0$, p, $m > 2$, and Ω is bounded domain of $\mathbb{R}^n (n \geq 1)$, with a smooth boundary $\partial\Omega$. For $b = 0$, it is well known that the damping term assures global existence for arbitrary initial data (see [3,5]). If $a = 0$, then the source term $bu|u|^{p-2}$ causes finite time blow up of solutions with negative initial energy (see [1,4,6,7]).

The interaction between the damping and the source terms was first considered by Levine [6,7] in linear damping case ($m = 2$). He showed that solutions with negative initial energy blow up in finite time. Recently Georgiev and Todorova [2] extended Levine's result to the nonlinear case, where the damping term is given by $au_t|u_t|^{m-2}$, $m > 2$. In their work, the authors introduced a different method than the famous concavity method and determined suitable relations between m and p, for which there is global existence or alternatively finite time blow up. Precisely, they showed that solutions with negative energy continue to exist globally 'in time' if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative.

This result has been lately generalized to an abstract setting and to unbounded domains by Levine and Serrin [8] and Levine *et al.* [9]. In these papers, the authors showed that no solution with negative initial energy can be extended on $[0, \infty)$ if $p > m$ and proved several noncontinuation theorems. This generalization allowed them also to apply their noncontinuation results to quasilinear situations, of which the result of [2] is a particular case. As they pointed out in [8], their result improved the result of [2] by not requiring that the initial energy be sufficiently negative but on the other hand the result of [2] showed that the noncontinuation implies a blow up. Messaoudi [11] used a slightly different technique to prove the same result of [2] without imposing the condition that the initial energy is sufficiently negative.

Vitillaro [12] combined the arguments in [2] and [8] to extend these results to situations where the damping is nonlinear and the solution has positive initial energy.

In this work we show that any solution of (1) with sufficiently negative initial energy blows up in finite time. We shall start by stating a local result, which is considered to be standard for such problems (see [10]).

THEOREM 1.1 *Suppose that $m \geq 2$, $p > 2$, and*

$$p \leq 2\frac{n-1}{n-2}, \quad n \geq 3. \tag{2}$$

Assume further that

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \tag{3}$$

Then the problem (1) *has a unique local solution*

$$u \in C\big((0,T); H_0^1(\Omega)\big), \quad u_t \in C\big([0,T); L^2(\Omega)\big) \cap L^m(\Omega \times (0,T)), \tag{4}$$

T is small.

2 MAIN RESULT

In this section, we show that the solution (4) blows up in finite time if $p > m$ and $-E(0)$ is sufficiently large, where

$$E(t) := \frac{1}{2}\int_\Omega [u_t^2 + |\nabla u|^2](x,t)dx - \frac{b}{p}\int_\Omega |u(x,t)|^p dx. \tag{5}$$

LEMMA 2.1 *Suppose that* (2) *holds. Then there exists a positive constant* $C > 1$ *depending on* Ω *only such that*

$$\|u\|_p^s \leq C(\|\nabla u\|_2^2 + \|u\|_p^p) \tag{6}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p$.

Proof If $\|u\|_p \leq 1$ then $\|u\|_p^s \leq \|u\|_p^2 \leq C\|\nabla u\|_2^2$ by Sobolev embedding theorems. If $\|u\|_p > 1$ then $\|u\|_p^s \leq \|u\|_p^p$. Therefore (6) follows.
We set

$$H(t) := -E(t)$$

and use, throughout this paper, C to denote a generic positive constant depending on a, m, p, and Ω only. As a result of (5), (6), and the lemma, we have

COROLLARY 2.2 *Let the assumptions of the lemma hold. Then we have*

$$\|u\|_p^s \leq C\Big(\big|H(t)\big| + \|u_t\|_2^2 + \|u\|_p^p\Big) \tag{7}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p$.

THEOREM 2.3 *Let the conditions of the Theorem* 1.1 *be fulfilled. Assume further that* $p > m$ *and* $-E(0)$ *is large enough then the solution* (4) *blows up in finite time.*

Proof
We multiply Eq. (1.1) by u_t and integrate over Ω to get

$$E'(t) = -a\left[\int_\Omega \left|u_t(x,t)\right|^m dx + \int_\Omega \left|u_t(x,t)\right|^2 dx\right], \tag{8}$$

for any regular solution of (1). This identity remains valid for solutions (4) by a simple density argument. So we have

$$0 < H(0) \le H(t) \le \frac{b}{p}\|u\|_p^p. \tag{9}$$

We then define

$$L(t) := e^{at}H^{1-\alpha}(t) + \varepsilon e^{at}\int_\Omega uu_t(x,t)dx \tag{10}$$

for ε small to be chosen late and

$$0 < \alpha \le \min\left\{\frac{(p-2)}{2p}, \frac{(p-m)}{p(m-1)}\right\}. \tag{11}$$

By taking a derivative of (10) and using Eq. (1) we obtain

$$\begin{aligned} L'(t) :=\ & e^{at}(1-\alpha)H^{-\alpha}(t)H'(t) + ae^{at}H^{1-\alpha}(t) + \varepsilon e^{at}\int_\Omega [u_t^2 - |\nabla u|^2](x,t)dx \\ & + \varepsilon b e^{at}\int_\Omega |u(x,t)|^p dx - a\varepsilon\int_\Omega |u_t|^{m-2}u_t u(x,t)dx. \end{aligned} \tag{12}$$

We then exploit Young's inequality

$$XY \le \frac{\delta^r}{r}X^r + \frac{\delta-q}{q}Y^q, \qquad X, Y, \ge 0, \quad \delta > 0, \quad \frac{1}{r} + \frac{1}{q} = 1$$

with $r=m$ and $q=m/(m-1)$ to estimate the last term in (12) as follows

$$\int_{\Omega} |u_t|^{m-1}|u|dx \leq \frac{\delta^m}{m}\|u\|_m^m + \frac{m-1}{m}\delta^{-m/(m-1)}\|u_t\|_m^m$$

which yields, by substitution in (12),

$$\begin{aligned} L'(t) \geq e^{at}\left[(1-\alpha)H^{-\alpha}(t)H'(t) - a\frac{m-1}{m}\varepsilon\delta^{-m/(m-1)}\|u_t\|_m^m\right] \\ + \varepsilon e^{at}\left[pH(t) + \frac{p}{2}\int_{\Omega}[u_t^2 + |\nabla u|^2](x,t)dx\right] \\ + \varepsilon e^{at}\int_{\Omega}[u_t^2 - |\nabla u|^2](x,t)dx - \varepsilon a e^{at}\frac{\delta^m}{m}\|u\|_m^m, \quad \forall \delta > 0. \end{aligned} \tag{13}$$

By noting that $H'(t) \geq a\|u_t\|_m^m$, we get

$$\begin{aligned} L'(t) \geq e^{at}\left[(1-\alpha)H^{-\alpha}(t) - \frac{m-1}{m}\varepsilon\delta^{-m/(m-1)}\right]H'(t) \\ + \varepsilon e^{at}\left(\frac{p}{2}-1\right)\int_{\Omega}|\nabla u|^2(x,t)dx + \varepsilon e^{at}pH(t) \\ + \varepsilon e^{at}\left(\frac{p}{2}+1\right)\int_{\Omega}u_t^2(x,t)dx - \varepsilon a e^{at}\frac{\delta^m}{m}\|u\|_m^m, \quad \forall \delta > 0. \end{aligned} \tag{14}$$

We then take δ so that $\delta^{-m/(m-1)} = kH^{-\alpha}(t)$, for large k to be specified later, and substitute in (14) to arrive at

$$\begin{aligned} L'(t) \geq e^{at}\left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)e^{at}\int_{\Omega}u_t^2(x,t)dx \\ + \varepsilon\left(\frac{p}{2}-1\right)e^{at}\int_{\Omega}|\nabla u|^2(x,t)dx + \varepsilon e^{at}\left[pH(t) - \frac{k^{1-m}}{m}aH^{\alpha(m-1)}(t)\|u\|_m^m\right]. \end{aligned} \tag{15}$$

By exploiting (9) and the inequality $\|u\|_m^m \leq C\|u\|_p^m$, we obtain

$$H^{\alpha(m-1)}(t)\|u\|_m^m \leq \left(\frac{b}{p}\right)^{\alpha(m-1)} C\|u\|_p^{m+\alpha p(m-1)},$$

hence (15) yields

$$L'(t) \geq e^{at}\left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)e^{at}\int_{\Omega} u_t^2(x,t)dx$$
$$+ \varepsilon\left(\frac{p}{2}-1\right)e^{at}\int_{\Omega} |\nabla u|^2(x,t)dx$$
$$+ \varepsilon e^{at}\left[pH(t) - \frac{k^{1-m}}{m}a\left(\frac{b}{p}\right)^{\alpha(m-1)} C\|u\|_p^{m+\alpha p(m-1)}\right]. \tag{16}$$

We then use Corollary 2.2 and (11), for $s = m + \alpha p(m-1) < p$, to deduce from (16)

$$L'(t) \geq e^{at}\left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t)$$
$$+ \varepsilon\left(\frac{p}{2}+1\right)e^{at}\int_{\Omega} u_t^2(x,t)dx + \varepsilon\left(\frac{p}{2}-1\right)e^{at}\int_{\Omega} |\nabla u|^2(x,t)dx$$
$$+ \varepsilon e^{at}\left[pH(t) - \frac{k^{1-m}}{m}\alpha\left(\frac{b}{p}\right)^{\alpha(m-1)} C\left\{H(t) + \|u_t\|_2^2 + \|u\|_p^p\right\}\right]. \tag{17}$$

At this point, we choose k large enough so that (17) becomes

$$L'(t) \geq e^{at}\left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t) + Ce^{at}\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p\right]. \tag{18}$$

We then pick $\varepsilon > 0$ small enough so that $(1-\alpha) - \varepsilon k(m-1)/m \geq 0$ and $L(0) > 0$. Therefore (18) takes the form

$$L'(t) \geq Ce^{at}\left[H(t) + \|u_t\|_2^2 + \|u\|_p^p\right]; \tag{19}$$

hence

$$L(t) \geq L(0) > 0, \quad \forall t \geq 0. \tag{20}$$

Next we estimate the second term in (10) as follows

$$\left|\int_\Omega uu_t(x,t)dx\right| \le \|u\|_2\|u_t\|_2 \le C\|u\|_p\|u_t\|_2$$

which implies

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \le C\|u\|_p^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}.$$

Again Young's inequality gives

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \le C\Big[\|u\|_p^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\Big], \tag{21}$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\alpha)$, to get $\mu/(1-\alpha) = 2/(1-2\alpha) \le p$ by (11). Therefore (21) becomes

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \le C\Big[\|u\|_p^s + \|u_t\|_2^2\Big],$$

where $s = 2/(1-2\alpha) \le p$. By using Corollary 2.2 we obtain

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)} \le \Gamma\Big[H(t) + \|u\|_p^p + \|u_t\|_2^2\Big], \quad \forall t \ge 0. \tag{22}$$

Consequently we have

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(e^{at}H^{1-\alpha}(t) + \varepsilon e^{at}\int_\Omega uu_t(x,t)dx\right)^{1/(1-\alpha)} \\ &\le 2^{1(1-\alpha)}e^{at/(1-\alpha)}\left(H(t) + \left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\alpha)}\right) \\ &\le Ce^{at/(1-\alpha)}\Big(H(t) + \|u\|_p^p + \|u_t\|_2^2\Big). \end{aligned} \tag{23}$$

We then combine (19) and (23), to arrive at

$$L'(t) \ge Ce^{-\alpha at/(1-\alpha)}L^{1/(1-\alpha)}(t) \tag{24}$$

A simple integration of (24) over $(0, t)$ yields

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - C(1 - e^{-\alpha a t/(1-\alpha)})} \tag{25}$$

Therefore by choosing the initial data so that $L^{\alpha/(1-\alpha)}(0) > 1/C$, $L(t)$ blows up in a time

$$T^* \leq \frac{(1-\alpha)\ln[C/(C - L^{-\alpha/(1-\alpha)}(0))]}{\alpha a} \tag{26}$$

Remark 2.1 The estimate (26) shows that the larger $L(0)$ is, the quicker the blow up takes place.

Remark 2.2 By following the steps of the proof of Theorem 2.3 closely, one can easily see that this result holds for $1 < m < p$, $p > 2$, Therefore this method is a unified one to both linear and nonlinear damping cases.

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References

[1] Ball, J. (1977). Remarks on blow up and nonexistence theorems for nonlinear evolutions equations. *Quart. J. Math. Oxford*, **28**(2), 473–486.
[2] Georgiev, V. and Todorova, G. (1994). Existence of solutions of the wave equation with nonlinear damping and source terms. *J. Diff. Eqns.*, **109**(2), 295–308.
[3] Haraux, A. and Zuazua, E. (1988). Decay estimates for some semilinear damped hyperbolic problems. *Arch. Rational Mech. Anal.*, **150**, 191–206.
[4] Kalantarov, V.K. and Ladyzhenskaya, O.A. (1978). The ocurence of collapse for quasilinear equations of parabolic and hyperbolic type. *J. Soviet Math.*, **10**, 53–70.
[5] Kopackova, M. (1989). Remarks on bounded solutions of a semilinear dissipative hyperbolic equation. *Comment. Math. Univ. Carolin*, **30**(4), 713–719.
[6] Levine, H.A. (1974). Instability and nonexistence of global solutions of nonlinear wave equation of the form $Pu_{tt} = Au + F(u)$, *Trans. Amer. Math. Soc.*, **192**, 1–21.
[7] Levine, H.A. (1974). Some additional remarks on the nonexistence of global solutions to nonlinear wave equation. *SIAM. J. Mach. Anal.*, **5**, 138–146.
[8] Levine, H.A. and Serrin, J. (1997). A global nonexistence theorem for quasilinear evolution equation with dissipation. *Arch. Rational Mech. Anal.*, **137**, 341–361.

[9] Levine, H.A. and Ro Park, S. (1998). Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation. *J. Math. Anal. Appl.*, **228**, 181–205.
[10] Lions, J.L. (1969). Quelques methods de resolution des problemes aux limites nonlinearies. *Dunond Gautier-villars Paris*.
[11] Messaoudi, S.A. (2001). Blow up in a nonlinearly damped wave equation, *Math. Nach.*, **231**, 105–111.
[12] Vittilaro, E. (1999). Global nonexistence theorems for a class of evolution equations with dissipation. *Arch. Rational Mech. Anal.*, **149**, 155–182.