Formation of singularities in solutions of a quasilinear strictly hyperbolic system.

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Abstract

We consider a special type of strictly hyperbolic systems and show that the gradient of the solution develops singularities in finite time.
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1 Introduction.

In a previous work [5], we considered the following one-dimensional quasilinear wave equation

$$w_{tt}(x,t) = \sigma\left(\frac{w_t(x,t)}{w_x(x,t)}\right) w_{xx}(x,t) \tag{1.1}$$

and showed that, for well chosen initial data, the classical solutions develop singularities in finite time. In the present work we prove a similar result for a strictly hyperbolic system, which can be regarded as a relative generalization of (1.1). More precisely we study the system

$$\begin{cases} u_t(x,t) = a\left(\dfrac{u(x,t)}{v(x,t)}\right) v_x(x,t) \\ v_t(x,t) = b\left(\dfrac{u(x,t)}{v(x,t)}\right) u_x(x,t) \end{cases} \tag{1.2}$$

where a subscript denotes a partial derivative with respect to the relevant variable; $x \in I = (0,1)$, and $t > 0$.

It is well known that, generally, classical solutions for hyperbolic systems break down in finite time even for smooth and small initial data. For instance Lax [7] and MacCamy and Mizel [12] studied the system for a depending on v only and $b \equiv 1$. They showed that clasical solutions blow up in finite time even if the initial data are smooth and small. In his work Lax required that $a^{'} > 0$; whereas MacCamy and Mizel allowed $a^{'}$ to change sign. Note that, in this particular case, the system is reduced to the well known nonlinear wave equation. For systems with dissipation the situation is different. If the initial data are smooth and small enough, then the effect of the damping term dominates the nonlinearity of the elastic response and global existence can be obtained. This has been established by Nishida [16] Li, Zhou, and Kong [10] and many others. In the other hand if the initial data are large then the nonliear elastic response destabilizes the solution and forces the gradient to blow up in finite time. This has been showed by Slemrod [18] and Kosinski [6](See also [13], [11], and [17]).

It is also worth mentioning that classes of semilinear hyperbolic systems have been recently treated by many authors and existence, as well as, nonexistence results have been established. (See for instance [1], [3]. [8], [9], [14], and [15]).

This paper is divided into two sections. In the first one we state without proof a local existence theorem. In the second one we establish our main result.

2 Local Existence.

We consider the following problem

$$u_t(x,t) = a\left(\frac{u(x,t)}{v(x,t)}\right) v_x(x,t) \tag{2.1}$$

$$v_t(x,t) = b\left(\frac{u(x,t)}{v(x,t)}\right) u_x(x,t), \qquad \forall x \in I = (0,\ 1), \quad t \geq 0 \tag{2.2}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad \forall x \in I \tag{2.3}$$

$$u(0,t) = u(1,t) = 0,\ v_x(0,t) = v_x(1,t) = 0, \quad t \geq 0, \tag{2.4}$$

where a, b, u_0, and v_0 are functions satisfying

$$a(\xi) \geq c > 0, b(\xi) \geq c > 0, \quad \forall \xi \in \mathbb{R} \tag{2.5}$$

$$u_0 \in H^2(I) \cap H_0^1(I), \quad v_0 \in H^2(I), \tag{2.6}$$

$$v_0(x) \neq 0, \qquad \forall x \in [0,1].$$

Proposition. *Assume that a and b are C^2 functions satisfying (2.5) and let u_0 and v_0 be given and satisfying (2.6). Then the problem (2.1) – (2.4) has a unique local solution (u,v), defined on a maximal time interval $[0,T)$, with*

$$u, v \in C\left([0,T),\ \ H^2(I)\right) \cap C^1\left([0,T),\ H^1(I)\right). \tag{2.7}$$

This result can be proved by either using a classical energy argument [2] or the nonlinear semigroup theory [4].

Remark 2.1. u, v are in $C^1([0,1] \times [0,T))$ by well-known Sobolev embedding theorems.

Remark 2.2. The local existence can also be established for (2.5) holds only in a neighbourhood of zero. In this case, we consider initial data satisfying (2.6) with $||u_0/v_0||_\infty$ small enough.

3 Formation of singularities.

In this section we state and prove our main result. We first begin with a lemma that gives a uniform bound on $|u/v|$ in terms of the initial data.

Lemma. *Let a and b be as in the proposition. Then there exist initial data satisfying (2.6), for wich $|u(x,t)/v(x,t)|$ remains uniformly bounded on $I \times [0,T)$.*

Proof. We define the quantities :

$$r := \ln|v| + \int_0^{u/v} \alpha(\xi) d\xi, \tag{3.1}$$

$$s := \ln|v| - \int_0^{u/v} \beta(\xi) d\xi,$$

and the differential operators :

$$\partial_t := \frac{\partial}{\partial t} - \rho(\frac{u}{v})\frac{\partial}{\partial x}, \quad D_t := \frac{\partial}{\partial t} + \rho(\frac{u}{v})\frac{\partial}{\partial x}, \tag{3.2}$$

where

$$\alpha(\xi) = \frac{1}{\gamma(\xi)+\xi}, \beta(\xi) = \frac{1}{\gamma(\xi)-\xi}, \quad \rho = \sqrt{ab},\ \gamma = \sqrt{\frac{a}{b}}. \tag{3.3}$$

We remaind that, unless otherwise stated, $a, b, P, \alpha, \beta, \gamma$, and ρ are functions of (u/v). We then study the evolution of r on the backward characteristic

$$\partial_t r = r_t - \rho r_x \tag{3.4}$$

$$\begin{aligned}
&= \frac{v_t}{v} + \alpha\frac{u_t v - u v_t}{v^2} - \rho(\frac{v_x}{v} + \alpha\frac{u_x v - u v_x}{v^2}) \\
&= \frac{1}{v}[(1 - \frac{u}{v}\alpha)v_t - \alpha\rho u_x] + \frac{1}{v}[\alpha u_t - \rho(1 - \frac{u}{v}\alpha)v_x] \\
&= \frac{\alpha\rho}{v}[\frac{1}{b}v_t - u_x] + \frac{\alpha}{v}[u_t - a v_x] = 0.
\end{aligned}$$

Similar computations also yield

$$D_t s = 0. \tag{3.5}$$

Therefore as long as a smooth solution continues to exist and

$$\left|\frac{u}{v}\right| < \gamma\left(\frac{u}{v}\right), \tag{3.6}$$

r and s remain constant along backward and forward characteristics respectively; hence

$$||r||_\infty = ||r_0||_\infty, \quad ||s||_\infty = ||s_0||_\infty. \tag{3.7}$$

To establish (3.6), we need to choose the initial data in a conviennient way. To this end we note that, by (3.1), we have

$$r - s = \phi(\frac{u}{v}), \tag{3.8}$$

where

$$\phi(\tau) = 2\int_0^\tau \gamma(\xi)/\left(\gamma^2(\xi) - \xi^2\right) d\xi \tag{3.9}$$

is continuous and strictly monotone, at least in a neighbouhood of zero. So it admits a continuous inverse ψ near zero. By noting that $g(\xi) = \gamma^2(\xi) - \xi^2$ is continuous and $g(0) > \gamma^2(0) > 0$, one can choose $\varepsilon > 0$ such that $g(\xi) > \gamma^2(0)/2$, for all $|\xi| < \varepsilon$. We then pick $\delta > 0$ so that $|\psi(\xi)| < \varepsilon$, for all $|\xi| < \delta$; consequently as long as $(r - s) < \delta$, we have

$$\left|\frac{u}{v}\right| = |\psi(r-s)| < \varepsilon. \tag{3.10}$$

Therefore by choosing u_0 and v_0 so that

$$||r_0||_\infty + ||s_0||_\infty < \delta, \tag{3.11}$$

the relation (3.6) holds and the proof of the lemma is completed.

Next, we introduce

$$P = a^{'}b + ab^{'}$$

Theorem. *Let a and b be as in the proposition. Assume further that*

$$P(0) > 0. \tag{3.12}$$

Then there exist initial data u_0, v_0 satisfying (2.6), for which the solution of the problem (2.1) – (2.4) blows up in finite time.

Proof. We take an x-partial derivative of (3.4) to get

$$(\partial_t r)_x = r_{xt} - \rho r_{xx} - r_x\rho_x = 0 \tag{3.13}$$

which, in turn, implies

$$\partial_t r_x = r_x\rho_x = \rho^{'} r_x \frac{\partial}{\partial x}\left(\frac{u}{v}\right). \tag{3.14}$$

By using

$$r_x = \frac{v_x}{v} + \alpha\frac{\partial}{\partial x}\left(\frac{u}{v}\right), s_x = \frac{v_x}{v} - \beta\frac{\partial}{\partial x}\left(\frac{u}{v}\right) \tag{3.15}$$

and substituting in (3.14), we obtain

$$\partial_t r_x = \frac{P}{4a}[\gamma^2 - \frac{u^2}{v^2}]r_x^2 - \frac{P}{4a}[\gamma^2 - \frac{u^2}{v^2}]r_x s_x. \tag{3.16}$$

To handle the last term in (3.16), we set

$$W := \lambda(\frac{u}{v})r_x$$

and substitute in (3.16), to get

$$\partial_t W = \lambda\frac{P}{4a}[\gamma^2 - \frac{u^2}{v^2}]r_x^2 - \lambda\frac{P}{4a}[\gamma^2 - \frac{u^2}{v^2}]r_x s_x + r_x\lambda^{'}\partial_t\left(\frac{u}{v}\right). \tag{3.17}$$

By using the equations, we get the estimate

$$\partial_t\left(\frac{u}{v}\right) = \frac{v(u_t - \sqrt{ab}u_x) - u(v_t - \sqrt{ab}v_x)}{v^2} \tag{3.18}$$

$$= \frac{v(av_x - \sqrt{ab}u_x) - u(bu_x - \sqrt{ab}v_x)}{v^2} = \frac{(\gamma v_x - u_x)(bu + \sqrt{ab}v)}{v^2}.$$

At this point, we choose λ so that

$$-\lambda\frac{P}{4a}[\gamma^2 - \frac{u^2}{v^2}]r_x s_x + r_x\lambda^{'}\frac{(\gamma v_x - u_x)(bu + \sqrt{ab}v)}{v^2} = 0. \tag{3.19}$$

By noting that

$$s_x = \frac{1}{v}\frac{1}{\gamma - \frac{u}{v}}(\gamma v_x - u_x) \tag{3.20}$$

and substituting in (3.19) we arrive, by simple computations, at

$$\frac{\lambda^{'}}{\lambda} = \frac{(ab)'}{4ab}, \tag{3.21}$$

which yields, by direct integration,

$$\lambda = [ab]^{1/4}. \tag{3.22}$$

Consequently (3.17) reduces to

$$\partial_t W = \frac{P}{4\lambda a}[\gamma^2 - \frac{u^2}{v^2}]W^2. \tag{3.23}$$

Therefore W (hence r_x) blows up in a time, if we choose initial data obeying (3.11) with derivatives satisfying

$$\frac{v_0'}{v_0} + \alpha(\frac{u_0}{v_0})\frac{u_0'v_0 - u_0v_0'}{v_0^2} > 0. \tag{3.24}$$

Remark 3.1. Similar result can be established if $P(0) < 0$. In this case we consider the evolution of s_x on the forward characteristics.

Remark 3.2. Initial data satisfying (3.11) and (3.24) can be easily constructed. For example $v_0 \equiv 1$ and u_0 is as small as possible.

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