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LOCAL EXISTENCE AND BLOW UP IN NONLINEAR THERMOELASTICITY WITH SECOND SOUND

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ABSTRACT

In this work we establish a local existence and a blow up result for a multidimensional nonlinear system of thermoelasticity with second sound.

Key Words: Thermoelasticity; Second sound; Nonlinear source; Negative initial energy; Local existence; Blow up; Finite time

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1. INTRODUCTION

Results concerning existence, blow up, and asymptotic behaviors of smooth, as well as weak, solutions in classical thermoelasticity have been established by several authors over the past two decades. See in this regard Refs. [1–3,5,6,8–15,18,20].

For thermoelasticity with second sound, global existence of smooth solutions for the one-dimensional case, has been established by Tarabek.[21] In his work, the author used the usual energy argument to prove his result. Saouli[18] used the nonlinear semigroup theory presented by Kato[4] to prove a local existence result for a system similar to the one considered in Ref. [21].

Concerning the asymptotic behavior, Racke[16] discussed lately the one-dimensional situation and established exponential decay results for several initial boundary value problems. In particular he showed that, for small enough initial data, classical solutions of a certain nonlinear problem decay exponentially to the equilibrium state. Regarding the multi-dimensional case $(n = 2, 3)$ Racke[17] established an existence result for the following n-dimensional problem

$$\begin{aligned}
&u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla\,\mathrm{div}\,u + \beta\nabla\theta = 0\\
&\theta_t + \gamma\,\mathrm{div}\,q + \delta\,\mathrm{div}\,u_t = 0\\
&\tau q_t + q + \kappa\nabla\theta = 0, \quad x\in\Omega, t>0\\
&u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0, \quad x\in\Omega\\
&u = \theta = 0, \quad x\in\partial\Omega, t\geq 0,
\end{aligned} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$, $u = u(x,t)\in\mathbb{R}^n$ is the displacement vector, $\theta = \theta(x,t)$ is the difference temperature, $q = q(x,t)\in\mathbb{R}^n$ is the heat flux vector, and $\mu,\lambda,\beta,\gamma,\delta,\tau,\kappa$ are positive constants, where μ,λ are Lame moduli and τ is the relaxation time, a small parameter compared to the others. In particular if $\tau = 0$, (1.1) reduces to the system of classical thermoelasticity, in which the heat flux is given by Fourier's law instead of Cattaneo's law. He also proved, under the conditions $\mathrm{rot}\,u = \mathrm{rot}\,q = 0$, an exponential decay result for (1.1). This result is extended to the radially semmetric solution, as it is only a special case.

In this paper we are concerned with the nonlinear problem

$$\begin{aligned}
&u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla\mathrm{div}u + \beta\nabla\theta = |u|^{p-2}u\\
&\theta_t + \gamma\,\mathrm{div}\,q + \delta\,\mathrm{div}\,u_t = 0\\
&\tau q_t + q + \kappa\nabla\theta = 0, \quad x\in\Omega, \quad t>0\\
&u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0, \quad x\in\Omega\\
&u = \theta = 0, \quad x\in\partial\Omega, \quad t\geq 0,
\end{aligned} \tag{1.2}$$

for $p > 2$. This is a similar problem to (1.1) with a nonlinear source term competing with the damping factor. We will establish a local existence result and show that solutions with negative energy blow up in finite time.

This work generalizes the one in Refs. [8,9] to thermoelasticity with second sound. This paper is organized as follows: in section two we establish the local existence. In section three the blow up result is proved.

2. LOCAL EXISTENCE

In this section, we establish a local existence result for (1.2) under a suitable condition on p. First we establish an existence result for a related linear problem

$$\begin{aligned}
&u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla\operatorname{div} u + \beta\nabla\theta = f\\
&\theta_t + \gamma\operatorname{div} q + \delta\operatorname{div} u_t = 0\\
&\tau q_t + q + \kappa\nabla\theta = 0, \quad x\in\Omega, \quad t>0\\
&u(.,0)=u_0, \quad u_t(.,0)=u_1, \quad \theta(.,0)=\theta_0, \quad q(.,0)=q_0, \quad x\in\Omega\\
&u=\theta=0, \quad x\in\partial\Omega, t\ge 0.
\end{aligned} \tag{2.1}$$

For this purpose we introduce the following spaces

$$\begin{aligned}
&\Pi := \left[H_0^1(\Omega)\cap H^2(\Omega)\right]^n \times \left[H_0^1(\Omega)\right]^n \times H_0^1(\Omega)\times D\\
&D := \left\{q\in[L^2(\Omega)]^n/\operatorname{div} q\in L^2(\Omega)\right\}\\
&H := \left[H_0^1(\Omega)\right]^n\times\left[L^2(\Omega)\right]^n\times L^2(\Omega)\times\left[L^2(\Omega)\right]^n
\end{aligned} \tag{2.2}$$

$$\Lambda := \max_{0\le t\le T}\left\{\|(u,u_t,\theta,q)(.,t)\|_\Pi^2 + \|(u_t,u_{tt},\theta_t,q_t)(.,t)\|_H^2\right\} \tag{2.3}$$

$$\Lambda_0 := \|(u_0,u_1,\theta_0,q_0)\|_\Pi^2 + \|(u_1,u_2,\theta_1,q_1)\|_H^2, \tag{2.4}$$

where

$$\begin{aligned}
&u_2 = \mu\Delta u_0 + (\mu+\lambda)\nabla\operatorname{div} u_0 - \beta\nabla\theta_0 + f(x,0)\\
&\theta_1 = -\gamma\operatorname{div} q_0 - \delta\operatorname{div} u_1\\
&q_1 = -[q_0+\kappa\nabla\theta_0]/\tau.
\end{aligned} \tag{2.5}$$

Lemma 2.1. *Assume that $f\in C^1([0,T);L^2(\Omega))^n$. Then given any initial data $(u_0,u_1,\theta_0,q_0)\in\Pi$, the problem* (2.1) *has a unique strong solution satisfying*

$$(u,u_t,\theta,q)\in C^1([0,T);\Pi)\cap C([0,T);H). \tag{2.6}$$

Moreover we have

$$\Lambda \le \Gamma\Lambda_0 + \Gamma T\max_{0\le t\le T}\left\{\|f(.,t)\|_2^2 + \|f_t(.,t)\|_2^2\right\}, \tag{2.7}$$

where Γ is a constant depending on $\mu,\lambda,\beta,\gamma,\delta,\kappa,\tau$ only.

Proof. The existence of solutions satisfying (2.6) is a direct result of Theorem 2.2 of Ref. [17]. To establish (2.7), we multiply (2.1) by $u_t, \beta\theta/\delta$, $\beta\gamma q/(\delta\kappa)$ respectively and integrate over $\Omega \times (0,t)$ to get

$$\frac{1}{2}\int_\Omega \left[|u_t|^2 + \mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2 + \frac{\beta}{\delta}|\theta|^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q|^2\right](x,t)\,dx$$
$$= \frac{1}{2}\int_\Omega \left[|u_1|^2 + \mu|\nabla u_0|^2 + (\lambda+\mu)(\operatorname{div} u_0)^2 + \frac{\beta}{\delta}|\theta_0|^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q_0|^2\right](x)\,dx$$
$$+ \int_0^t \int_\Omega f(x,s).u_t(x,s)\,dx\,ds. \tag{2.8}$$

To obtain estimates on terms involving higher order derivatives, we apply the difference operator

$$\Delta_h w(x,t) := w(x+h,t) - w(x,t), \quad x \in \Omega, \quad t \in [0,T), \quad 0 < h < T-t$$

to the Eq. (2.1). By multiplying the resulting equations by $\Delta_h u_t$, $\beta\Delta_h\theta/\delta$, $\beta\gamma\Delta_h q/(\delta\kappa)$ respectively, integrating over $\Omega \times (0,t)$, using integration by parts, dividing by h^2, and letting h go to zero we arrive at

$$\frac{1}{2}\int_\Omega \left[|u_{tt}|^2 + \mu|\nabla u_t|^2 + (\lambda+\mu)(\operatorname{div} u_t)^2 + \frac{\beta}{\delta}|\theta_t|^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q_t|^2\right](x,t)\,dx$$
$$= \frac{1}{2}\int_\Omega \left[|u_2|^2 + \mu|\nabla u_1|^2 + (\lambda+\mu)(\operatorname{div} u_1)^2 + \frac{\beta}{\delta}|\theta_1|^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q_1|^2\right](x)\,dx$$
$$+ \int_0^t \int_\Omega f_t(x,s).u_{tt}(x,s)\,dx\,ds. \tag{2.9}$$

By combining (2.8), (2.9), the (2.1), and using Cauchy-Schwarz inequality, (2.7) is established.

Lemma 2.2. *Assume that*

$$2 < p \le \frac{2(n-3)}{n-4}, \quad n \ge 5 \tag{2.10}$$

and $v \in C([0,T); H^2(\Omega))^n \cap (C^1([0,T); H^1(\Omega))^n$. *Then* $f = |v|^{p-2}v$ *satisfies*

$$\int_\Omega |f(x,t)|^2 dx \le C\|v\|_{H^2}^{2p-2}, \qquad \int_\Omega |f_t(x,t)|^2 dx \le C\|v_t\|_{H^1}^2 \|v\|_{H^2}^{2p-4}, \tag{2.11}$$

where C *is a constant depending on* Ω *and* p *only.*

The proof is trivial. We only use the embedding of Sobolev spaces in the L^q spaces.

Remark 2.1. For $n \le 4$, (2.11) remains valid without imposing (2.10).

Theorem 2.3. *Assume that* (2.10) *holds. Then given any* $(u_0, u_1, \theta_0, \ q_0) \in \Pi$, *the problem* (1.2) *has a unique strong solution satisfying*. (2.6), *for* T *small enough*.

Proof. For $M > 0$ large and $T > 0$, we define a class of functions $Z(M, T)$ which consists of all functions (w, ϕ, ξ) satisfying (2.6), the initial conditions of (1.2), and

$$\max_{0 \le t \le T} \left\{ \|(u, u_t, \theta, q)(.,t)\|_{\Pi}^2 + \|(u_t, u_{tt}, \theta_t, q_t)(.,t)\|_H^2 \right\} \le M^2. \tag{2.12}$$

$Z(M, T)$ is nonempty if M is large enough. This follows from the trace theorem.[7] We also define the map F by $(u, \theta, q) := F(w, \phi, \xi)$, where (u, θ, q) is the unique solution of the linear problem

$$\begin{aligned}
&u_{tt} - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u + \beta \nabla \theta = |v|^{p-2} v \\
&\theta_t + \gamma \operatorname{div} q + \delta \operatorname{div} u_t = 0 \\
&\tau q_t + q + \kappa \nabla \theta = 0, \quad x \in \Omega, \quad t > 0 \\
&u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0, \quad x \in \Omega \\
&u = \theta = 0, \quad x \in \partial\Omega, \quad t \ge 0
\end{aligned} \tag{2.13}$$

since $|v|^{p-2} v \in [L^2(\Omega)]^n$ by virtue of (2.10). We would like to show, for M sufficiently large and T sufficiently small, that F is a contraction from $Z(M, T)$ into itself.

By using (2.7) and (2.11) we get

$$\begin{aligned}
&\max_{0 \le t \le T} \left\{ \|(u, u_t, \theta, q)(.,t)\|_{\Pi}^2 + \|(u_t, u_{tt}, \theta_t, q_t)(.,t)\|_H^2 \right\} \\
&\qquad \le \Gamma \Lambda_0 + \Gamma C T \max_{0 \le t \le T} \left\{ \|v\|_{H^2}^{2p-2} + \|v_t\|_{H^1}^2 \|v\|_{H^2}^{2p-4} \right\} \le \Gamma \Lambda_0 + \Gamma C T M^{2p-2}.
\end{aligned}$$

By choosing M large enough and T sufficiently small, (2.12) is established; hence $(u, \theta, q) \in Z(M, T)$. So F maps $Z(M, T)$ into itself.

Next we prove that F is a contraction. For this aim we equip $Z(M, T)$ with the complete[1] metric

$$\begin{aligned}
&d((v^m, \phi^m, \xi^m), (v^l, \phi^l, \xi^l)) \\
&\quad = \sqrt{\max_{0 \le t \le T} \|(v^m - v^l, v_t^m - v_t^l, \theta^m - \theta^l, q^m - q^l)(.,t)\|_H^2}
\end{aligned}$$

[1]The completeness of the metric d follows from the weak * precompactness of bounded sets in $L^\infty([0, T); L^2(\Omega))$ and sequential weak lower semicontinuity of norms in these spaces (see Ref. [20]).

and set

$$
\begin{aligned}
&U := u^m - u^l, \quad \Theta := \theta^m - \theta^l, \quad Q = q^m - q^l \\
&V := v^m - v^l, \quad \Phi := \phi^m - \phi^l, \quad \zeta = \xi^m - \xi^l
\end{aligned}
$$

where $(u^m, \theta^m, q^m) = F(v^m, \phi^m, \xi^m)$ and $(u^l, \theta^l, q^l) = F(v^l, \phi^l, \xi^l)$. It is straightforward to see that (U, Θ, Q) satisfies

$$
\begin{aligned}
&U_{tt} - \mu\Delta U - (\mu + \lambda)\nabla \operatorname{div} U + \beta\nabla\Theta = |v^m|^{p-2}v^m - |v^l|^{p-2}v^l \\
&\Theta_t + \gamma \operatorname{div} Q + \delta \operatorname{div} U_t = 0 \\
&\tau Q_t + Q + \kappa\nabla\Theta = 0, \quad x \in \Omega, \quad t > 0 \\
&U(.,0) = U_t(.,0) = \Theta(x,0) = Q(.,0) = 0, \quad x \in \Omega \\
&U = \Theta = 0, \quad x \in \partial\Omega, \ t \geq 0.
\end{aligned}
\tag{2.14}
$$

We multiply (2.14) by U_t, $\beta\Theta/\delta$, $\beta\gamma Q/(\delta\kappa)$ respectively and integrate over $\Omega \times (0, t)$ to get

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \left[|U_t|^2 + |\nabla U|^2 + (\lambda + \mu)(\operatorname{div} U)^2 + \frac{\beta}{\delta}|\Theta|^2 + \frac{\gamma\beta\tau}{\delta\kappa}|Q|^2\right](x,t)\,dx \\
&\quad \leq \int_0^t \int_\Omega \| v^m|^{p-2}v^m - |v^l|^{p-2}v^l \| U_t|(x,s)\,dx\,ds \\
&\quad \leq C\int_0^t \|U_t\|_2 \|V\|_{H^1}\left\{\|v^m\|_{H^2}^{p-2} + \|v^l\|_{H^2}^{p-2}\right\}(.,s)\,ds.
\end{aligned}
\tag{2.15}
$$

Therefore (2.15) yields

$$
d((u^m, \theta^m, q^m), (u^l, \theta^l, q^l)) \leq \Gamma T M^{p-2} d((v^m, \phi^m, \xi^m), (v^l, \phi^l, \xi^l)). \tag{2.16}
$$

By choosing T so small that $\Gamma T M^{p-2} < 1$, the estimate (2.16) shows that F is a contraction. The contraction mapping theorem then guarantees the existence of a unique (u, θ, q) satisfying $(u, \theta, q) = F(u, \theta, q)$. Obviously it is the unique solution of (1.2). The proof is completed.

3. BLOW UP RESULT

In this section we show that the solution (2.6) blows up in finite time if $E(0) < 0$, where

$$
\begin{aligned}
E(t) := &-\frac{1}{p}\int_\Omega |u(x,t)|^p\,dx + \frac{1}{2}\int_\Omega \left(|u_t|^2 + \mu|\nabla u|^2\right)(x,t)\,dx \\
&+ \frac{1}{2}\int_\Omega \left((\mu + \lambda)(\operatorname{div} u)^2 + \frac{\beta}{\delta}\theta^2 + \frac{\gamma\beta\tau}{\delta\kappa}|q|^2\right)(x,t)\,dx.
\end{aligned}
\tag{3.1}
$$

Lemma 3.1. *Suppose that*

$$2 < p \leq \frac{2n}{n-2}, \quad n \geq 3. \tag{3.2}$$

Then there exists a positive constant $C > 1$ *depending on* Ω *and* p *only such that*

$$\|u\|_p^s \leq C\left(\|\nabla u\|_2^2 + \|u\|_p^p\right) \tag{3.3}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p$.

Proof. If $\|u\|_p \leq 1$ then $\|u\|_p^s \leq \|u\|_p^2 \leq C\|\nabla u\|_2^2$ by Sobolev embedding theorems. If $\|u\|_p > 1$ then $\|u\|_p^s \leq \|u\|_p^p$. Therefore (3.3) follows.

We set

$$H(t) := -E(t) \tag{3.4}$$

and use, throughout this paper, C to denote a generic positive constant depending on Ω and p only. As a result of (3.1)–(3.4), we have

Corollary 3.2. *Assume that* (3.2) *holds. Then*

$$\begin{aligned}\|u\|_p^s \leq C\Bigg\{&\left(1+\frac{2}{p\mu}\right)\|u\|_p^p - \frac{2}{\mu}H(t) - \frac{1}{\mu}\|u_t\|_2^2 \\ &- \left(1+\frac{\lambda}{\mu}\right)\|\operatorname{div} u\|_2^2 - \frac{\beta}{\delta\mu}\|\theta\|_2^2 - \frac{\gamma\beta\tau}{\delta\kappa}\|q\|_2^2\Bigg\},\end{aligned} \tag{3.5}$$

for any $u \in (H_0^1(\Omega))^n$ *and* $2 \leq s \leq p$.

Theorem 3.3. *Assume that* (3.2) *holds. Assume further that*

$$p(p+2) > \frac{\beta\tau\delta}{\kappa\gamma}. \tag{3.6}$$

Then for any initial data in Π *satisfying*

$$E(0) < 0, \tag{3.7}$$

the solution (2.6) *blows up in finite time.*

Remark 3.1. The condition (3.6) is 'physically' reasonable due to the very small value of τ. For instance in Ref. [16], for the isotropic silicon and a medium temperature of 300K we have

$$\beta \approx 391.62\left[\frac{m^2}{s^2K}\right], \quad \tau \approx 10^{-12}[s], \quad \delta \approx 163.82[K],$$

$$\gamma \approx 5.99 \times 10^{-7}\left[\frac{ms^2K}{kg}\right], \quad \kappa \approx 148\left[\frac{W}{mK}\right]$$

consequently we get

$$\frac{\beta\tau\delta}{\kappa\gamma} \approx 72.367 \times 10^{-7}$$

So (3.6) is satisfied for any $p > 2$.

Remark 3.2. If $\tau = 0$, then (1.2) reduces to the classical system of thermoelasticity and the blow up result takes place without condition (3.6). This is exactly what was proven in Refs. [8,9]. See also remarks by the end of Ref. [10].

Proof. We multiply Eq. (1.2) by $-u_t$, $-\beta\theta/\delta$, $-\beta\gamma q/\delta\tau$ respectively and integrate over Ω, using integration by parts, and add equalities to get

$$H'(t) = \frac{\gamma\beta}{\delta\kappa}\|q\|_2^2 \geq 0, \quad \forall t \in [0, T); \tag{3.8}$$

consequently we get

$$0 < H(0) \leq H(t), \qquad \forall t \in [0, T), \tag{3.9}$$

by virtue of (3.1) and (3.4). We then introduce

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega \left[u.u_t + \frac{\beta\tau}{\kappa} u.q \right](x, t)\, dx \tag{3.10}$$

for ε small to be chosen later and

$$\alpha = \frac{(p-2)}{(2p)}. \tag{3.11}$$

By taking a derivative of (3.10) and using Eq. (1.2) we obtain

$$\begin{aligned} L'(t) = {} & (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\left\{ \|u\|_p^p + \|u_t\|_2^2 - \mu\|\nabla u\|_2^2 - (\mu+\lambda)\|\operatorname{div} u\|_2^2 \right\} \\ & - \varepsilon\frac{\beta}{\kappa}\int_\Omega u.q\, dx + \varepsilon\frac{\beta\tau}{\kappa}\int_\Omega u_t.q\, dx. \end{aligned} \tag{3.12}$$

We then use (3.1) and (3.4) to substitute for $\|u\|_p^p$; hence (3.12) takes the form

$$\begin{aligned} L'(t) = {} & (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p}{2}+1\right)\|u_t\|_2^2 + \mu\varepsilon\left(\frac{p}{2}-1\right)\|\nabla u\|_2^2 \\ & + \varepsilon(\mu+\lambda)\left(\frac{p}{2}-1\right)\|\operatorname{div} u\|_2^2 + \varepsilon\frac{p\beta}{2\delta}\|\theta\|_2^2 + \varepsilon\frac{p\beta\gamma\tau}{2\kappa\delta}\|q\|_2^2 \\ & + \varepsilon p H(t) - \varepsilon\frac{\beta}{\kappa}\int_\Omega u.q\, dx + \varepsilon\frac{\beta\tau}{\kappa}\int_\Omega u_t.q\, dx. \end{aligned} \tag{3.13}$$

We then exploit Young's inequality to estimate the last two terms in (3.13) as follows

$$\left|\int_\Omega u_t.q\,dx\right| \le \frac{a}{2}\|u_t\|_2^2 + \frac{1}{2a}\|q\|_2^2, \quad \forall a > 0$$
$$\int_\Omega u.q\,dx \le \frac{b}{2}\|u\|_2^2 + \frac{1}{2b}\|q\|_2^2, \quad \forall b > 0.$$

Thus (3.13) yields

$$\begin{aligned} L'(t) \ge{}& (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p+2}{2} - \frac{a\beta\tau}{2\kappa}\right)\|u_t\|_2^2 + \mu\varepsilon\left(\frac{p}{2}-1\right)\|\nabla u\|_2^2 \\ &+ \varepsilon(\mu+\lambda)\left(\frac{p}{2}-1\right)\|\operatorname{div} u\|_2^2 + \varepsilon\frac{p\beta}{2\delta}\|\theta\|_2^2 + \varepsilon\frac{\beta\tau}{2\kappa}\left(\frac{p\gamma}{\delta} - \frac{1}{a}\right)\|q\|_2^2 \\ &+ \varepsilon pH(t) - \varepsilon\frac{\beta}{\kappa}\left(\frac{b}{2}\|q\|_2^2 + \frac{1}{2b}\|u\|_2^2\right). \end{aligned} \tag{3.14}$$

At this point we choose a so that

$$A_1 := \frac{p+2}{2} - \frac{a\beta\tau}{2\kappa} > 0, \qquad A_2 := \frac{\beta\tau}{2\kappa}\left(\frac{p\gamma}{\delta} - \frac{1}{a}\right) > 0.$$

This is possible by virtue of (3.6); hence (3.14) becomes

$$\begin{aligned} L'(t) \ge{}& (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon A_1\|u_t\|_2^2 + \varepsilon A_2\|q\|_2^2 + \varepsilon A_3\|\nabla u\|_2^2 \\ &+ \varepsilon A_4\|\operatorname{div} u\|_2^2 + \varepsilon A_5\|\theta\|_2^2 + \varepsilon pH(t) - \varepsilon\frac{\beta}{\kappa}\left(\frac{b}{2}\|q\|_2^2 + \frac{1}{2b}\|u\|_2^2\right), \end{aligned} \tag{3.15}$$

where A_1–A_5 are strictly positive constants depending only on $p, \beta, \gamma, \delta, \kappa, , \lambda, \mu, \tau$. We also set $b = 2M\gamma H^{-\alpha}(t)/\delta$; for M a constant to be determined; hence (3.15) gives

$$\begin{aligned} L'(t) \ge{}& [(1-\alpha) - \varepsilon M]H^{-\alpha}(t)H'(t) + \varepsilon A_1\|u_t\|_2^2 + \varepsilon A_2\|q\|_2^2 + \varepsilon A_3\|\nabla u\|_2^2 \\ &+ \varepsilon A_4\|\operatorname{div} u\|_2^2 + \varepsilon A_5\|\theta\|_2^2 + \varepsilon pH(t) - \frac{C\varepsilon}{4M}H^{\alpha}(t)\|u\|_p^2, \end{aligned} \tag{3.16}$$

where C, here and in the sequel, is a positive generic constant depending on $\Omega, p, \beta, \gamma, \delta, \kappa, \lambda, \mu, \tau$ only. We then use $H(t) \le \|u\|_p^p/p$ to get, from (3.16),

$$\begin{aligned} L'(t) \ge{}& [(1-\alpha) - \varepsilon M]H^{-\alpha}(t)H'(t) + \varepsilon A_1\|u_t\|_2^2 + \varepsilon A_2\|q\|_2^2 + \varepsilon A_3\|\nabla u\|_2^2 \\ &+ \varepsilon A_4\|\operatorname{div} u\|_2^2 + \varepsilon A_5\|\theta\|_2^2 + \varepsilon pH(t) - \frac{C\varepsilon}{4M}\left(\frac{1}{p}\right)\|u\|_p^{2+\alpha p}. \end{aligned} \tag{3.17}$$

Since $2+\alpha p<p$ we exploit (3.5) to obtain, from (3.17),

$$\begin{aligned} L'(t) \geq{}& [(1-\alpha)-\varepsilon M]H^{-\alpha}(t)H'(t)+\varepsilon\left(A_1+\frac{C}{M}\right)\|u_t\|_2^2 \\ &+\varepsilon\left(A_2+\frac{C}{M}\right)\|q\|_2^2+\varepsilon A_3\|\nabla u\|_2^2+\varepsilon\left(A_4+\frac{C}{M}\right)\|\operatorname{div} u\|_2^2 \\ &+\varepsilon\left(A_5+\frac{C}{M}\right)\|\theta\|_2^2+\varepsilon\left(p+\frac{C}{M}\right)H(t)-\frac{C\varepsilon}{M}\left(1+\frac{2}{p\mu}\right)\|u\|_p^p. \end{aligned} \tag{3.18}$$

At this point, we choose M large enough so that the coefficients of the terms in (3.18) are strictly positive; hence we get

$$\begin{aligned} L'(t) \geq{}& [(1-\alpha)-\varepsilon M]H^{-\alpha}(t)H'(t) \\ &+\varepsilon A_0\big[H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\|\operatorname{div} u\|_2^2+\|q\|_2^2+\|u\|_p^p\big], \end{aligned} \tag{3.19}$$

where $A_0>0$ is the minimum of these coefficients. Once M is fixed (hence A_0), we pick ε small enough so that $(1-\alpha)-\varepsilon M\geq 0$ and

$$L(0)=H^{1-\alpha}(0)+\varepsilon\int_\Omega u_0.(u_1+\frac{\beta\tau}{\kappa}q)(x)\,dx>0.$$

Therefore (3.19) leads to

$$L'(t)\geq A_0\varepsilon\big[H(t)+\|u_t\|_2^2+\|q\|_2^2+\|u\|_p^p\big]. \tag{3.20}$$

Consequently we have

$$L(t)\geq L(0)>0, \quad \forall t\geq 0.$$

Next we estimate

$$\left|\int_\Omega uu_t(x,t)\,dx\right|\leq C\|u\|_2\|u_t\|_2\leq C\|u\|_p\|u_t\|_2,$$

which implies

$$\left|\int_\Omega uu_t(x,t)\,dx\right|^{1/(1-\alpha)}\leq C\|u\|_p^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}.$$

Again Young's inequality gives us

$$\left|\int_\Omega uu_t(x,t)\,dx\right|^{1/(1-\alpha)}\leq C\Big[\|u\|_p^{r/(1-\alpha)}+\|u_t\|_2^{s/(1-\alpha)}\Big], \tag{3.21}$$

for $1/r+1/s=1$. We take $s=2(1-\alpha)$, to get $r/(1-\alpha)=2/(1-2\alpha)=p$ by virtue of (3.11). Therefore (3.21) becomes

$$\left|\int_\Omega uu_t(x,t)\,dx\right|^{1/(1-\alpha)}\leq C\big[\|u\|_p^p+\|u_t\|_2^2\big], \quad \forall t\geq 0. \tag{3.22}$$

Similarly we have

$$\left|\int_\Omega uq(x,t)\,dx\right|^{1/(1-\alpha)} \le C\big[\|u\|_p^p + \|u_t\|_2^2\big], \quad \forall t \ge 0. \tag{3.23}$$

Finally by noting that

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon\int_\Omega u\left(u_t + \frac{\beta\tau}{\kappa}q\right)(x,t)\,dx\right)^{1/(1-\alpha)} \\ &\le C\left(H(t) + \left|\int_\Omega uu_t(x,t)\,dx\right|^{1/(1-\alpha)} + \left|\int_\Omega uq(x,t)\,dx\right|^{1/(1-\alpha)}\right) \\ &\le C\big[H(t) + \|u\|_p^p + \|u_t\|_2^2 + \|q\|_2^2\big], \quad \forall t \ge 0. \end{aligned}$$

and combining it with (3.20), (3.22), (3.23) we obtain

$$L'(t) \ge a_0 L^{1/(1-\alpha)}(t), \quad \forall t \ge 0 \tag{3.24}$$

where a_0 is a positive constant depending on εA_0 and C. A simple integration of (3.24) over $(0,t)$ then yields

$$L^{(p-2)/(p+2)}(t) \ge \frac{1}{L^{-(p-2)/2}(0) - a_0 t(p-2)/2}.$$

Therefore $L(t)$ blows up in a time

$$T^* \le \frac{1-\alpha}{\alpha a_0 [L(0)]^{(p-2)/(p+2)}}. \tag{3.25}$$

Remark 3.3. The estimate (3.25) shows that the larger $L(0)$ is the quicker the blow up takes place.

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