Blow up in the Cauchy problem for a nonlinearly damped wave equation

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Abstract

In this paper we consider the Cauchy problem for the nonlinearly damped wave equation with nonlinear source

$$u_{tt} - \Delta u + au_t|u_t|^{m-2} = bu|u|^{p-2},$$

$p > m$. We prove that given any time $T > 0$, there exist always initial data with sufficiently negative initial energy, for which the solution blows up in time $\leq T$. This result improves an earlier one by Todorova [11].
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1 Introduction

In this paper we are concerned with the following initial value problem

$$\begin{array}{c} u_{tt} - \Delta u + au_t|u_t|^{m-2} = bu|u|^{p-2}, \qquad x \in \mathbb{R}^n, \qquad t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \mathbb{R}^n, \end{array} \tag{1.1}$$

where $a, b > 0$, and $p, m > 2$. For the initial boundary value problem, it is well known that if $b = 0$ then the damping term $au_t|u_t|^{m-2}$ assures global existence for arbitrary initial data (see [4], [6]). If $a = 0$ then the source term $bu|u|^{p-2}$ causes finite time blow up of solutions with negative initial energy (see [1], [5], [7], [8]).

The interaction between the damping and the source terms, for the IBVP, was first considered by Levine [7], [8] in the linear damping case ($m = 2$). He showed that solutions with negative initial energy cannot be global in time. Georgiev and Todorova [3] extended Levine's result to the nonlinear damping case ($m > 2$). In their work, the authors introduced a different method and determined suitable relations between m and p, for which there is global existence or alternatively finite time blow up. Precisely they showed that solutions with negative energy continue to exist globally 'in time' if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative. This result was improved by Levine and Serrin [9] and Messaoudi [10]. In their work, Levine and Serrin [9] treated an abstract problem, showed that no solution with negative energy can be extended on $[0, \infty)$ if $p > m$, and proved several noncontinuation theorems. This generalization allowed them to apply their results to quasilinear situations, of which problem (1.1) is a particular case.

For solutions with positive initial energy, we mention the blow up results of Todorova [12] and Vitillaro [13]. In his paper, Vitillaro also studied an abstract problem and established many existence and nonexistence results to the semilinear, as well as quasilinear, cases.

In all above results, the boundedness of the domain played an essential role because of the usage of the injection of the L^p spaces. In a recent work, Todorova [11] treated the Cauchy problem (1.1) for compactly supported initial data. She showed that the weak solution of (1.1) exists globally ' in time ' if $m \geq p$ and it blows up in finite time for any initial data with negative energy if $p > m > np/(n+p+1)$. When $m < np/(n+p+1)$ the solution blows up if the initial energy is sufficiently negative and $\int u_0 u_1 \geq 0$. She also established a similar result for (1.1) with a source of the form $bu|u|^{p-2} - q^2(x)u$ under a suitable condition on $q(x)$.

In this work, we show that the condition $\int u_0 u_1 \geq 0$ is unnecessary and the result can be proved without it. We do not consider the same functional as in [11] and show that given any time $T > 0$, there exist initial data, with sufficiently negative energy, for wich the solution blows up in a time $t^* \leq T$. We first state a local result (See [3] and [11]).

Theorem 1. *Suppose that* $m > 2$, $p > 2$, *and*

$$p \leq \frac{2\,(n-1)}{n-2}, \quad n \geq 3. \tag{1.2}$$

Then for any initial data

$$(u_0, u_1) \in H_0^1(\mathbb{R}^n) \text{ x } L^2(\mathbb{R}^n) \tag{1.3}$$

with $suppu\,(u_0) \cup supp\,(u_1) \subset B_R(0)$, *then there exists* $T_m > 0$ *such that problem (1.1) has a unique local solution*

$$u \in C\left([0,\ T_m);\ H_0^1(\mathbb{R}^n)\right), \quad u_t \in C\left([0,\ T_m);\ L^2(\mathbb{R}^n)\right) \cap L^m\left(\mathbb{R}^n \text{ x } (0,\ T_m)\right). \tag{1.4}$$

Remark 1.1 The exponent (1.2) is the cut for p needed to establish the local existence. See relation (2.6) in [3].

2 Main Result.

In this section we show that the solution (1.4) blows up in finite time if $p > m$ and the initial energy

$$E_0 := \frac{1}{2}\int [u_1^2 + |\nabla u_0|^2](x)dx - \frac{b}{p}\int |u_0(x)|^p dx \tag{2.5}$$

is negative enough.

Lemma 2.1. *Suppose that (1.2) holds and $p > 1$. Then there exists a positive constant C depending on n and p only such that*

$$||u||_p \leq C\,(L)^{1/p-1/p^*}\,||\nabla u||_2, \quad p^* = 2n/(n-2), \tag{2.6}$$

for any $u \in H^1(\mathbb{R}^n)$, with supp$\,(u) \subset B_L(0)$.

Proof. If $u \in H^1(\mathbb{R}^n)$ then $u \in L^{p^*}(\mathbb{R}^n)$, and $||u||_{p^*} \leq C_1||\nabla u||_2$, , where C_1 is a constant depending on n (See theorem IX.9 of [2]). But $||u||_p \leq C_2(L)^{1/p-1/p^*}$ $||u||_{p^*}$, for any $p \leq p^*$, where C_2 is a constant depending on n, p, and p^*. Therefore (2.2) follows.

Remark 2.2. For the case $n < 3$, we have

$$||u||_p \leq C\,(L)^{1/p+1/2}\,||\nabla u||_2, \quad n = 1 \tag{2.7}$$

by theorem IX.12 of [2] and

$$||u||_p \leq C||\nabla u||_2, \quad n = 2 \tag{2.8}$$

by corollary IX.11 of [2], C is a constant depending on n and p only.

Remark 2.3. Without loss of generality, L and R (below) are taken larger than or equal to one.

Lemma 2.4. *Suppose that $2 \leq s \leq p$ and (1.2) holds if $n \geq 3$. Then there exists a positive constant C depending on n and p only such that*

$$||u||_p^s \leq C\,(L)^{1/p+1/2}\left(||\nabla u||_2^2 + ||u||_p^p\right) \tag{2.9}$$

for any $u \in H^1(\mathbb{R}^n)$, with supp$\,(u) \subset B_L(0)$.

Proof. If $||u||_p \leq 1$ then $||u||_p^s \leq ||u||_p^2$. From (2.2), it follows that $||u||_p^s \leq C\,(L)^{1/p+1/2}\,||\nabla u||_2^2$. If $||u||_p > 1$ then $||u||_p^s \leq ||u||_p^p$. Therefore (2.5) follows.

We set

$$H(t) := -\frac{1}{2}\int [u_t^2 + |\nabla u|^2](x,t)dx + \frac{b}{p}\int |u(x,t)|^p dx. \tag{2.10}$$

As a consequence of (2.5), (2.6), of fact that supp$\,(u_0) \cup$ supp$\,(u_1) \subset B_R(0)$, and of finite speed propagation, we have

Corollary 2.5. *Let the assumptions of theorem 1 hold. Then the solution defined by (1.4) satisfies*

$$||u||_p^s \leq C\,(R+t)^{1/p+1/2}\left(|H(t)| + ||u_t||_2^2 + ||u||_p^p\right). \tag{2.11}$$

Theorem 2. *Suppose that $p > m > 2$ and (1.2) holds if $n \geq 3$. Then for any $R \geq 1$ and $T > 0$ there exists $M > 0$ such that for initial data (u_0, u_1) satisfying (1.3), with supp$\,(u_0) \cup$ supp$\,(u_1) \subset B_R(0)$, and*

$$E_0 < -M, \tag{2.12}$$

the solution (1.4) blows up in a time $t^* \leq T$.

Remark 2.5. Note that we do not require $\int u_0 u_1 \geq 0$.

Proof.

We multiply equation (1.1) by $-u_t$ and integrate over $\mathbb{R}^n$ to get

$$H'(t) = a\int |u_t(x,t)|^m dx,$$

for almost every t in $[0,T)$ since $H(t)$ is absolutely continuous (see [2]). So we have

$$0 < -E_0 = H(0) \le H(t) \le \frac{b}{p}||u||_p^p, \tag{2.13}$$

for every t in $[0,T)$, by virtue of (2.6). We then define

$$J(t) := H^{1-\alpha}(t) + \varepsilon\int uu_t(x,t)dx \tag{2.14}$$

for ε small to be chosen later and

$$0 < \alpha \le min\left\{\frac{(p-2)}{2p}, \frac{(p-m)}{p(m-1)}\right\}. \tag{2.15}$$

By taking the derivative of (2.10) and using equation (1.1) we obtain

$$J'(t) := (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int [u_t^2 - |\nabla u|^2](x,t)dx \tag{2.16}$$

$$+\varepsilon b\int |u(x,t)|^p dx - a\varepsilon\int |u_t|^{m-2}u_t u(x,t)dx.$$

We then exploit Young's inequality

$$XY \le \frac{\delta^r}{r}X^r + \frac{\delta^{-q}}{q}Y^q, \qquad X,Y \ge 0,\ \delta > 0, \qquad \frac{1}{r} + \frac{1}{q} = 1$$

for $r = m$ and $q = m/(m-1)$ to estimate the last term in (2.12) as follows

$$\int |u_t|^{m-1}|u|dx \le \frac{\delta^m}{m}||u||_m^m + \frac{m-1}{m}\delta^{-m/(m-1)}||u_t||_m^m.$$

A substitution in (2.12) yields

$$J'(t) \ge \left[(1-\alpha)H^{-\alpha}(t) - \frac{m-1}{m}\varepsilon\delta^{-m/(m-1)}\right]H'(t) + \varepsilon\int [u_t^2 - |\nabla u|^2](x,t)dx$$

$$+\varepsilon\left[pH(t) + \frac{p}{2}\int_\Omega [u_t^2 + |\nabla u|^2](x,t)dx\right] - \varepsilon a\frac{\delta^m}{m}||u||_m^m, \qquad \forall\delta > 0. \tag{2.17}$$

Of course (2.13) remains valid even if δ is time dependent since the integral is taken over the x variable. Therefore by taking δ so that $\delta^{-m/(m-1)} = kH^{-\alpha}(t)$, for large k to be specified later, and substituting in (2.13) we arrive at

$$J'(t) \ge \left[(1-\alpha) - \frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t) + \varepsilon(\frac{p}{2}+1)\int u_t^2(x,t)dx \tag{2.18}$$

$$+\varepsilon(\frac{p}{2}-1)\int|\nabla u|^2(x,t)dx+\varepsilon\left[pH(t)-\frac{k^{1-m}}{m}aH^{\alpha(m-1)}(t)||u||_m^m\right].$$

By exploiting (2.9) and the inequality

$$||u||_m^m\leq C||u||_p^m(R+t)^{n(p-m)/p},$$

we obtain

$$H^{\alpha(m-1)}(t)||u||_m^m\leq C\left(\frac{b}{p}\right)^{\alpha(m-1)}(R+t)^{n(p-m)/p}||u||_p^{m+\alpha p(m-1)};$$

hence (2.14) yields

$$J'(t)\geq\left[(1-\alpha)-\frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t)+\varepsilon(\frac{p}{2}+1)\int u_t^2(x,t)dx \tag{2.19}$$

$$+\varepsilon(\frac{p}{2}-1)\int|\nabla u|^2(x,t)dx$$

$$+\varepsilon\left[pH(t)-C\frac{k^{1-m}}{m}a\left(\frac{b}{p}\right)^{\alpha(m-1)}(R+t)^{n(p-m)/p}||u||_p^{m+\alpha p(m-1)}\right].$$

We then use corollary 2.5, for $s=m+\alpha p(m-1)\leq p$, to deduce from (2.15)

$$J'(t)\geq\left[(1-\alpha)-\frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t)+\varepsilon(\frac{p}{2}+1)\int u_t^2(x,t)dx \tag{2.20}$$

$$+\varepsilon(\frac{p}{2}-1)\int|\nabla u|^2(x,t)dx$$

$$+\varepsilon\left[pH(t)-C_1k^{1-m}(R+T)^{\beta}\left\{H(t)+||u_t||_2^2+||u||_p^p\right\}\right],\forall t\leq T,$$

where $C_1=Ca\left(\frac{b}{p}\right)^{\alpha(m-1)}/m.$ and $\beta=n(p-m)/p+1/p+1/2$. At this point, we choose k large enough so that (2.16) takes the forrm

$$J'(t)\geq\left[(1-\alpha)-\frac{m-1}{m}\varepsilon k\right]H^{-\alpha}(t)H'(t)+\varepsilon\gamma\left[H(t)+||u_t||_2^2+||u||_p^p\right], \tag{2.21}$$

where $\gamma>0$ is a constant depending on C_1, k, and $(R+T)^{\beta}$. Once k is fixed (hence γ), we pick ε small enough so that $(1-\alpha)-\varepsilon k(m-1)/m\geq 0$ and

$$J(0)=H^{1-\alpha}(0)+\varepsilon\int u_0u_1(x)dx>0.$$

Therefore (2.17) becomes

$$J'(t)\geq\gamma\varepsilon\left[H(t)+||u_t||_2^2+||u||_p^p\right]. \tag{2.22}$$

Consequently we have

$$J(t)\geq J(0)>0,\qquad\forall t\leq T.$$

Next we estime

$$|\int uu_t(x,t)dx| \le ||u||_2||u_t||_2 \le C(R+T)^{n(p-2)/2p}||u||_p||u_t||_2$$

which implies

$$|\int uu_t(x,t)dx|^{1/(1-\alpha)} \le C(R+T)^{\nu}||u||_p^{1/(1-\alpha)}||u_t||_2^{1/(1-\alpha)},$$

where $\nu = n(p-2)/2p(1-\alpha)$. Again Young's inequality gives us

$$|\int uu_t(x,t)dx|^{1/(1-\alpha)} \le C(R+T)^{\nu}\left[||u||_p^{\mu/(1-\alpha)} + ||u_t||_2^{\theta/(1-\alpha)}\right], \tag{2.23}$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\alpha)$, to get $\mu/(1-\alpha) = 2/(1-2\alpha) \le p$ by (2.9). Therefore (2.19) becomes

$$|\int uu_t(x,t)dx|^{1/(1-\alpha)} \le C(R+T)^{\nu}\left[||u||_p^{s} + ||u_t||_2^{2}\right],$$

where $s = 2/(1-2\alpha) \le p$. By using corollary 2.5 we obtain

$$|\int uu_t(x,t)dx|^{1/(1-\alpha)} \le C(R+T)^{\nu+1/p+1/2}\left[H(t) + ||u||_p^{p} + ||u_t||_2^{2}\right], \qquad \forall t \le T. \tag{2.24}$$

Finally note that

$$\begin{aligned} J^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon\int uu_t(x,t)dx\right)^{1/(1-\alpha)} \\ &\le 2^{1/(1-\alpha)}\left(H(t) + |\int uu_t(x,t)dx|^{1/(1-\alpha)}\right) \\ &\le C(R+T)^{\nu+1/p+1/2}\left[H(t) + ||u||_p^{p} + ||u_t||_2^{2}\right], \qquad \forall t \le T. \end{aligned} \tag{2.25}$$

A combination of (2.18) and (2.21) then yields

$$J'(t) \ge \Gamma J^{1/(1-\alpha)}(t), \quad \forall t \le T, \tag{2.26}$$

where $\Gamma = \varepsilon\gamma/C(R+T)^{\nu+1/p+1/2}$. A direct integration over $(0, \quad t)$ gives

$$J^{\alpha/(1-\alpha)}(t) \ge \frac{1}{J^{-\alpha/(1-\alpha)}(0) - \alpha\Gamma t/(1-\alpha)}, \quad \forall t \le T. \tag{2.27}$$

Therefore (2.23) shows that for M, introduced in (2.8), large enough J blows up in a time $t^* \le T$.

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