



ELSEVIER

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

J. Math. Anal. Appl. 320 (2006) 902–915

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation

Salim A. Messaoudi

Mathematical Sciences Department, KFUPM, Dhahran 31261, Saudi Arabia

Received 17 December 2004

Available online 1 September 2005

Submitted by J. Lavery

Abstract

In this paper, we consider the nonlinear viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau + u_t |u_t|^{m-2} = u |u|^{p-2}$$

with initial conditions and Dirichlet boundary conditions. For nonincreasing positive functions g and for $p > m$, we prove that there are solutions with positive initial energy that blow up in finite time.

© 2005 Elsevier Inc. All rights reserved.

Keywords: Blow-up; Finite time; Hyperbolic; Nonlinear damping; Positive initial energy; Viscoelastic

1. Introduction

In this paper, we are concerned with the initial-boundary-value problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau + u_t |u_t|^{m-2} = u |u|^{p-2}, & \text{in } \Omega \times (0, \infty), \\ u(x, t) = 0, & x \in \partial\Omega, t \geq 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega. \end{cases} \quad (1.1)$$

E-mail address: messaoud@kfupm.edu.sa.

where Ω is a bounded domain of \mathbb{R}^n ($n \geq 1$) with a smooth boundary $\partial\Omega$, $p > 2$, $m \geq 1$, and g is a positive function. In the absence of the viscoelastic term (that is, if $g = 0$), the equation in (1.1) reduces to the nonlinearly damped wave equation

$$u_{tt} - \Delta u + u_t |u_t|^{m-2} = u |u|^{p-2}.$$

This equation has been extensively studied by many mathematicians. It is well known that in the further absence of the damping mechanism $u_t |u_t|^{m-2}$, the source term $u |u|^{p-2}$ causes finite-time blow-up of solutions with negative initial energy (see [1,9]). In contrast, in the absence of the source term, the damping term assures global existence for arbitrary initial data (see [8,10]). The interaction between the damping and source terms was first considered by Levine [11,12] for linear damping ($m = 2$). Levine showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [7] extended Levine's result to nonlinear damping ($m > 2$). In their work, the authors introduced a new method and determined relations between m and p for which there is global existence and other relations between m and p for which there is finite-time blow-up. Specifically, they showed that solutions with negative energy continue to exist globally if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative. Messaoudi [15] extended the blow-up result of [7] to solutions with only negative initial energy. For related results, we refer the reader to Levine and Serrin [13], Levine and Ro Park [14], Vitillaro [19], Yang [20] and Messaoudi and Said-Houari [18].

In the presence of the viscoelastic term ($g \neq 0$), Cavalcanti et al. [4] studied (1.1) for $m = 2$ and a localized damping mechanism $a(x)u_t$ ($a(x)$ null on a part of the domain). They obtained an exponential rate of decay by assuming that the kernel g is of exponential decay. This work was later improved by Cavalcanti et al. [6] and Berrimi and Messaoudi [2] using different methods. In related work, Cavalcanti et al. [3] studied solutions of

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau) \Delta u(\tau) d\tau - \gamma \Delta u_t = 0, \quad x \in \Omega, \quad t > 0,$$

for $\rho > 0$ and proved a global existence result for $\gamma \geq 0$ and an exponential decay result for $\gamma > 0$. This latter result was extended by Messaoudi and Tatar [16] to a situation where a source term is competing with the damping induced by $-\gamma \Delta u_t$ and the integral term. Also, Cavalcanti et al. [5] established an existence result and a decay result for viscoelastic problems with nonlinear boundary damping.

Concerning nonexistence, Messaoudi [17] showed that Todorova and Georgiev's results can be extended to (1.1) using the technique of [7] with a modification in the energy functional due to the different nature of the problems.

In this article, we improve our earlier result by adopting and modifying the method of [19]. In particular, we will show that there are solutions of (1.1) with positive initial energy that blow up in finite time.

We first state a local existence theorem that can be established by combining arguments of [4,7].

Theorem 1.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Let $m > 1, p > 2$ be such that

$$\max\{m, p\} \leq \frac{2(n-1)}{n-2}, \quad n \geq 3. \tag{1.2}$$

Let g be a C^1 function satisfying

$$1 - \int_0^\infty g(s) ds = l > 0. \tag{1.3}$$

Then problem (1.1) has a unique local solution

$$u \in C([0, T_m); H_0^1(\Omega)), \quad u_t \in C([0, T_m); L^2(\Omega)) \cap L^m(\Omega \times (0, T_m)), \tag{1.4}$$

for some $T_m > 0$.

Remark 1.1. Condition (1.2) is needed to establish the local existence result (see [4,7]). In fact under this condition, the nonlinearity in the source is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$. Condition (1.3) is necessary to guarantee the hyperbolicity and well-posedness of system (1.1).

Next we state our main result. For this purpose, we assume that g satisfies, in addition to (1.3), the inequalities

$$g(s) \geq 0, \quad g'(s) \leq 0, \\ \int_0^\infty g(s) ds < \frac{(p/2) - 1}{(p/2) - 1 + (1/2p)}. \tag{1.5}$$

Theorem 1.2. Let m and p be such that $m > 1, p > \max\{2, m\}$ and (1.2) holds. Assume further that g satisfies (1.3), (1.5). Then any solution of (1.1) with initial data satisfying (2.7) below blows up in finite time.

2. Proof of the blow-up result

In this section we prove our main result (Theorem 1.2). For this purpose we let B be the best constant of the Sobolev embedding $[H^1] \hookrightarrow [L^p]$ and $B_1 = B/l^{1/2}$. We set

$$\alpha = B_1^{-p/(p-2)}, \quad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)\alpha^2. \tag{2.1}$$

We also define

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s) ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_p^p, \tag{2.2}$$

where

$$(g \circ v)(t) = \int_0^t g(t - \tau) \|v(t) - v(\tau)\|_2^2 d\tau.$$

Lemma 2.1. Assume that (1.2), (1.3) and (1.5) hold. Let u be a solution of (1.1). Then $E(t)$ is nonincreasing, that is,

$$E'(t) \leq 0. \tag{2.3}$$

Proof. By multiplying Eq. (1.1) by u_t and integrating over Ω we obtain

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} |u_t|^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx \right\} \\ - \int_0^t g(t - \tau) \int_{\Omega} \nabla u_t(t) \cdot \nabla u(\tau) dx d\tau = - \int_{\Omega} |u_t|^m dx, \end{aligned} \tag{2.4}$$

for any regular solution. This result remains valid for weak solutions by a simple density argument. For the last term on the left side of (2.4) we have

$$\begin{aligned} & \int_0^t g(t - \tau) \int_{\Omega} \nabla u_t(t) \cdot \nabla u(\tau) dx d\tau \\ &= \int_0^t g(t - \tau) \int_{\Omega} \nabla u_t(t) \cdot [\nabla u(\tau) - \nabla u(t)] dx d\tau \\ & \quad + \int_0^t g(t - \tau) \int_{\Omega} \nabla u_t(t) \cdot \nabla u(t) dx d\tau \\ &= -\frac{1}{2} \int_0^t g(t - \tau) \frac{d}{dt} \int_{\Omega} |\nabla u(\tau) - \nabla u(t)|^2 dx d\tau \\ & \quad + \int_0^t g(\tau) \left(\frac{d}{dt} \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 dx \right) d\tau \\ &= -\frac{1}{2} \frac{d}{dt} \left[\int_0^t g(t - \tau) \int_{\Omega} |\nabla u(\tau) - \nabla u(t)|^2 dx d\tau \right] \\ & \quad + \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(\tau) \int_{\Omega} |\nabla u(t)|^2 dx d\tau \right] \end{aligned}$$

$$+ \frac{1}{2} \int_0^t g'(t - \tau) \int_{\Omega} |\nabla u(\tau) - \nabla u(t)|^2 dx d\tau - \frac{1}{2} g(t) \int_{\Omega} |\nabla u(t)|^2 dx d\tau. \tag{2.5}$$

Inserting (2.5) into (2.4), we obtain

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} |u_t|^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx \right\} \\ & + \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(t - \tau) \int_{\Omega} |\nabla u(\tau) - \nabla u(t)|^2 dx d\tau \right] \\ & - \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(\tau) \|\nabla u(t)\|^2 d\tau \right] \\ & = - \int_{\Omega} |u_t|^m dx + \frac{1}{2} \int_0^t g'(t - \tau) \int_{\Omega} |\nabla u(\tau) - \nabla u(t)|^2 dx d\tau \\ & - \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 \leq 0. \end{aligned} \tag{2.6}$$

This completes the proof. \square

Lemma 2.2. Assume that (1.2), (1.3) and (1.5) hold. Let u be a solution of (1.1) with initial data satisfying

$$E(0) < E_1, \quad \|\nabla u_0\|_2 > B_1^{-p/(p-2)}. \tag{2.7}$$

Then there exists a constant $\beta > B_1^{-p/(p-2)}$ such that

$$\left[\left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) \right]^{1/2} \geq \beta, \quad \forall t \in [0, T), \tag{2.8}$$

and

$$\|u\|_p \geq B_1 \beta, \quad \forall t \in [0, T). \tag{2.9}$$

Proof. We first note that, by (2.2), we have

$$\begin{aligned} E(t) & \geq \frac{1}{2} \left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) - \frac{1}{p} \|u\|_p^p \\ & \geq \frac{1}{2} \left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) - \frac{1}{p} B_1^p l^p \|\nabla u\|_2^p \end{aligned}$$

$$\begin{aligned}
 &\geq \frac{1}{2} \left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) \\
 &\quad - \frac{B_1^p}{p} \left[\left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) \right]^{p/2} \\
 &= \frac{1}{2} \zeta^2 - \frac{B_1^p}{p} \zeta^p = h(\zeta),
 \end{aligned} \tag{2.10}$$

where

$$\zeta = \left[\left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) \right]^{1/2}.$$

It is easy to verify that h is increasing for $0 < \zeta < \alpha$, decreasing for $\zeta > \alpha$, $h(\zeta) \rightarrow -\infty$ as $\zeta \rightarrow +\infty$, and

$$h(\alpha) = \left(\frac{1}{2} - \frac{1}{p} \right) B_1^{-2p/(p-2)} = E_1,$$

where α is given in (2.1). Therefore, since $E(0) < E_1$, there exists $\beta > \alpha$ such that $h(\beta) = E(0)$. If we set $\alpha_0 = \|\nabla u_0\|_2$ then, by (2.10), we have

$$h(\alpha_0) \leq E(0) = h(\beta).$$

Therefore, $\alpha_0 > \beta$.

To establish (2.8), we suppose by contradiction that

$$\left[\left(1 - \int_0^{t_0} g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t_0) \right]^{1/2} < \beta,$$

for some $t_0 > 0$. By the continuity of

$$\left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t),$$

we can choose t_0 such that

$$\left[\left(1 - \int_0^{t_0} g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t_0) \right]^{1/2} > \alpha.$$

Again, the use of (2.10) leads to

$$E(t_0) \geq h \left(\left[\left(1 - \int_0^{t_0} g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t_0) \right]^{1/2} \right) > h(\beta) = E(0).$$

This is impossible since $E(t) \leq E(0)$, for all $t \in [0, T)$. Hence (2.8) is established.

To prove (2.9), we exploit (2.2). We have

$$\frac{1}{2} \left[\left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) \right] \leq E(0) + \frac{1}{p} \|u\|_p^p.$$

Consequently, we obtain

$$\begin{aligned} \frac{1}{p} \|u\|_p^p &\geq \frac{1}{2} \left[\left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) \right] - E(0) \\ &\geq \frac{1}{2} \beta^2 - E(0) \\ &\geq \frac{1}{2} \beta^2 - h(\beta) = \frac{B_1^p}{p} \beta^p. \end{aligned} \tag{2.11}$$

The proof is complete. \square

Lemma 2.3. *Suppose that (1.2) holds. Then there exists a positive constant $C > 1$ such that*

$$\|u\|_p^s \leq C (\|\nabla u\|_2^2 + \|u\|_p^p) \tag{2.12}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

Proof. If $\|u\|_p \leq 1$ then $\|u\|_p^s \leq \|u\|_p^2 \leq C \|\nabla u\|_2^2$ by Sobolev embedding.

If $\|u\|_p > 1$ then $\|u\|_p^s \leq \|u\|_p^p$. Therefore, (2.12) follows. This completes the proof. \square

We set

$$H(t) = E_1 - E(t) \tag{2.13}$$

and use, throughout this paper, C to denote a generic positive constant depending on p and l only. As a result of (2.2), (2.12), and (2.13), we have

Lemma 2.4. *Let u be solution of (1.1). Assume that (1.2) holds. Then we have*

$$\|u\|_p^s \leq C \left(-H(t) - \|u_t\|_2^2 - (g \circ \nabla u)(t) + \|u\|_p^p \right), \quad \forall t \in [0, T), \tag{2.14}$$

for any $2 \leq s \leq p$.

Proof. Using (1.3) and (2.2), we note that

$$\begin{aligned} \frac{1}{2} (1-l) \|\nabla u\|_2^2 &\leq \frac{1}{2} \left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 \\ &\leq E(t) - \frac{1}{2} \|u_t\|_2^2 - \frac{1}{2} (g \circ \nabla u)(t) + \frac{1}{p} \|u\|_p^p \\ &\leq E_1 - H(t) - \frac{1}{2} \|u_t\|_2^2 - \frac{1}{2} (g \circ \nabla u)(t) + \frac{1}{p} \|u\|_p^p. \end{aligned} \tag{2.15}$$

Exploiting (2.1) and (2.9), simple calculations yield

$$E_1 \leq \frac{p-2}{2p} \|u\|_p^p. \tag{2.16}$$

Finally, a combination of (2.15) and (2.16) gives the desired result. \square

Proof of Theorem 1.2. Using (2.2), (2.3) and (2.13), we obtain

$$\begin{aligned} 0 < H(0) &\leq H(t) \\ &\leq E_1 - \frac{1}{2} \left[\|u_t\|_2^2 + \left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) \right] + \frac{1}{p} \|u\|_p^p \end{aligned}$$

and, from (2.8), we obtain

$$\begin{aligned} E_1 - \frac{1}{2} \left[\|u_t\|_2^2 + \left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) \right] \\ < E_1 - \frac{1}{2} \beta^2 = -\frac{1}{p} \beta^2 < 0, \quad \forall t \geq 0. \end{aligned} \tag{2.17}$$

Hence,

$$0 < H(0) \leq H(t) \leq \frac{1}{p} \|u\|_p^p, \quad \forall t \geq 0. \tag{2.18}$$

We define

$$L(t) := H^{1-\sigma}(t) + \varepsilon \int_{\Omega} u u_t dx, \tag{2.19}$$

for small ε to be chosen later and for

$$0 < \sigma \leq \min \left\{ \frac{(p-2)}{2p}, \frac{(p-m)}{p(m-1)} \right\}. \tag{2.20}$$

Taking a derivative of (2.19) and using Eq. (1.1), we obtain

$$\begin{aligned} L'(t) &= (1-\sigma)H^{-\sigma}(t) \left\{ \|u_t\|_m^m - \frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2 \right\} \\ &\quad + \varepsilon \int_{\Omega} [u_t^2 - |\nabla u|^2] dx + \varepsilon \int_0^t g(t-\tau) \int_{\Omega} \nabla u(t) \cdot \nabla u(\tau) dx d\tau \\ &\quad + \varepsilon \int_{\Omega} |u|^p dx - \varepsilon \int_{\Omega} |u_t|^{m-2} u_t u dx \\ &\geq (1-\sigma)H^{-\sigma}(t) \|u_t\|_m^m + \varepsilon \int_{\Omega} [u_t^2 - |\nabla u|^2] dx \end{aligned}$$

$$\begin{aligned}
 & + \varepsilon \int_{\Omega} |u|^p dx - \varepsilon \int_{\Omega} |u_t|^{m-2} u_t u dx + \varepsilon \int_0^t g(t - \tau) \|\nabla u(t)\|_2^2 d\tau \\
 & + \varepsilon \int_0^t g(t - \tau) \int_{\Omega} \nabla u(t) \cdot [\nabla u(\tau) - \nabla u(t)] dx d\tau.
 \end{aligned} \tag{2.21}$$

Using the Schwarz inequality, (2.21) takes on the form

$$\begin{aligned}
 L'(t) & \geq (1 - \sigma)H^{-\sigma}(t) \|u_t\|_m^m + \varepsilon \int_{\Omega} [u_t^2 - |\nabla u|^2] dx \\
 & + \varepsilon \int_{\Omega} |u|^p dx - \varepsilon \int_{\Omega} |u_t|^{m-2} u_t u dx \\
 & - \varepsilon \int_0^t g(t - \tau) \|\nabla u(t)\|_2 \|\nabla u(\tau) - \nabla u(t)\|_2 d\tau \\
 & + \varepsilon \int_0^t g(t - \tau) \|\nabla u(t)\|_2^2 d\tau.
 \end{aligned} \tag{2.22}$$

We now exploit Young’s inequality to estimate the fifth term on the right side of (2.22) and use (2.2) to substitute for $\int_{\Omega} |u(x, t)|^p dx$. Hence, (2.22) becomes

$$\begin{aligned}
 L'(t) & \geq (1 - \sigma)H^{-\sigma}(t) \|u_t\|_m^m + \varepsilon \int_{\Omega} u_t^2 dx - \varepsilon \left(1 - \int_0^t g(s) ds\right) \|\nabla u(t)\|_2^2 \\
 & + \varepsilon \left(p H(t) + \frac{p}{2} (g \circ \nabla u)(t) + \frac{p}{2} \|u_t\|_2^2 + \frac{p}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla u(t)\|_2^2 \right) \\
 & - \varepsilon \int_{\Omega} |u_t|^{m-2} u_t u(x, t) dx - \varepsilon \tau (g \circ \nabla u)(t) - \frac{\varepsilon}{4\tau} \int_0^t g(s) ds \|\nabla u(t)\|_2^2 \\
 & \geq (1 - \sigma)H^{-\sigma}(t) \|u_t\|_m^m + \varepsilon \left(1 + \frac{p}{2}\right) \int_{\Omega} u_t^2 dx + \varepsilon p H(t) \\
 & + \varepsilon \left(\frac{p}{2} - \tau\right) (g \circ \nabla u)(t) - \varepsilon \int_{\Omega} |u_t|^{m-2} u_t u dx \\
 & + \varepsilon \left(\left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\tau}\right) \int_0^{\infty} g(s) ds \right) \|\nabla u(t)\|_2^2,
 \end{aligned} \tag{2.23}$$

for some number τ with $0 < \tau < p/2$. Recalling (1.5), the estimate (2.23) reduces to

$$\begin{aligned}
 L'(t) \geq & (1 - \sigma)H^{-\sigma}(t)\|u_t\|_m^m + \varepsilon\left(1 + \frac{p}{2}\right) \int_{\Omega} u_t^2(x, t) dx \\
 & + \varepsilon p H(t) + \varepsilon a_1(g \circ \nabla u)(t) + \varepsilon a_2 \|\nabla u(t)\|_2^2 - \varepsilon \int_{\Omega} |u_t|^{m-2} u_t u \, dx, \quad (2.24)
 \end{aligned}$$

where

$$a_1 = \frac{p}{2} - \tau > 0, \quad a_2 = \left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\tau}\right) \int_0^\infty g(s) \, ds > 0.$$

To estimate the last term of (2.24), we again use Young’s inequality

$$XY \leq \frac{\delta^r}{r} X^r + \frac{\delta^{-q}}{q} Y^q, \quad X, Y, \geq 0, \quad \forall \delta > 0, \quad \frac{1}{r} + \frac{1}{q} = 1$$

with $r = m$ and $q = m/(m - 1)$. So we have

$$\int_{\Omega} |u_t|^{m-1} |u| \, dx \leq \frac{\delta^m}{m} \|u\|_m^m + \frac{m-1}{m} \delta^{-m/(m-1)} \|u_t\|_m^m,$$

which yields, by substitution in (2.24),

$$\begin{aligned}
 L'(t) \geq & \left[(1 - \sigma)H^{-\sigma}(t) - \frac{m-1}{m} \varepsilon \delta^{-m/(m-1)} \right] \|u_t\|_m^m \\
 & + \varepsilon \left(1 + \frac{p}{2}\right) \int_{\Omega} u_t^2(x, t) \, dx + \varepsilon a_1(g \circ \nabla u)(t) \\
 & + \varepsilon a_2 \|\nabla u(t)\|_2^2 + \varepsilon p H(t) - \varepsilon \frac{\delta^m}{m} \|u\|_m^m, \quad \forall \delta > 0. \quad (2.25)
 \end{aligned}$$

Of course (2.25) remains valid even if δ is time-dependant since the integral is taken over the x variable. Therefore, taking δ so that $\delta^{-m/(m-1)} = k H^{-\sigma}(t)$ for large k to be specified later and substituting in (2.25), we arrive at

$$\begin{aligned}
 L'(t) \geq & \left[(1 - \sigma) - \frac{m-1}{m} \varepsilon k \right] H^{-\sigma}(t) \|u_t\|_m^m + \varepsilon \left(\frac{p}{2} + 1\right) \int_{\Omega} u_t^2(x, t) \, dx \\
 & + \varepsilon a_1(g \circ \nabla u)(t) + \varepsilon a_2 \|\nabla u(t)\|_2^2 \\
 & + \varepsilon \left[p H(t) - \frac{k^{1-m}}{m} H^{\sigma(m-1)}(t) \|u\|_m^m \right]. \quad (2.26)
 \end{aligned}$$

Exploiting (2.18) and the inequality $\|u\|_m^m \leq C \|u\|_p^m$, we obtain

$$H^{\sigma(m-1)}(t) \|u\|_m^m \leq \left(\frac{1}{p}\right)^{\sigma(m-1)} C \|u\|_p^{m+\sigma p(m-1)}.$$

Hence, (2.26) yields

$$\begin{aligned}
 L'(t) \geq & \left[(1 - \sigma) - \frac{m - 1}{m} \varepsilon k \right] H^{-\sigma}(t) \|u_t\|_m^m \\
 & + \varepsilon \left(\frac{p}{2} + 1 \right) \int_{\Omega} u_t^2(x, t) \, dx + \varepsilon a_1 (g \circ \nabla u)(t) + \varepsilon a_2 \|\nabla u(t)\|_2^2 \\
 & + \varepsilon \left[p H(t) - \frac{k^{1-m}}{m} \left(\frac{1}{p} \right)^{\sigma(m-1)} C \|u\|_p^{m+\sigma p(m-1)} \right].
 \end{aligned} \tag{2.27}$$

We now use (2.20) and Lemma 2.4 with $s = m + \sigma p(m - 1) \leq p$ to deduce from (2.27) that

$$\begin{aligned}
 L'(t) \geq & \left[(1 - \sigma) - \frac{m - 1}{m} \varepsilon k \right] H^{-\sigma}(t) \|u_t\|_m^m \\
 & + \varepsilon \left(\frac{p}{2} + 1 \right) \int_{\Omega} u_t^2(x, t) \, dx + \varepsilon a_1 (g \circ \nabla u)(t) + \varepsilon a_2 \|\nabla u(t)\|_2^2 \\
 & + \varepsilon [p H(t) - C_1 k^{1-m} \{-H(t) - \|u_t\|_2^2 - (g \circ \nabla u)(t) + \|u\|_p^p\}] \\
 \geq & \left[(1 - \sigma) - \frac{m - 1}{m} \varepsilon k \right] H^{-\sigma}(t) \|u_t\|_m^m \\
 & + \varepsilon \left(\frac{p}{2} + 1 + C_1 k^{1-m} \right) \|u_t\|_2^2 + \varepsilon (a_1 + C_1 k^{1-m}) (g \circ \nabla u)(t) \\
 & + \varepsilon a_2 \|\nabla u(t)\|_2^2 + \varepsilon (p + C_1 k^{1-m}) H(t) - \varepsilon C_1 k^{1-m} \|u\|_p^p,
 \end{aligned} \tag{2.28}$$

where $C_1 = (1/p)^{\sigma(m-1)} C/m$. Noting that

$$H(t) \geq \frac{1}{p} \|u\|_p^p - \frac{1}{2} \|u_t\|_2^2 - \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{2} (g \circ \nabla u)(t)$$

and writing $p = 2a_3 + (p - 2a_3)$, where $a_3 < \min\{a_1, a_2, p/2\}$, the estimate (2.28) yields

$$\begin{aligned}
 L'(t) \geq & \left[(1 - \sigma) - \frac{m - 1}{m} \varepsilon k \right] H^{-\sigma}(t) \|u_t\|_m^m \\
 & + \varepsilon \left(\frac{p}{2} + 1 + C_1 k^{1-m} - a_3 \right) \|u_t\|_2^2 + \varepsilon (a_1 + C_1 k^{1-m} - a_3) (g \circ \nabla u)(t) \\
 & + \varepsilon (a_2 - a_3) \|\nabla u(t)\|_2^2 + \varepsilon (p - 2a_3 + C_1 k^{1-m}) H(t) \\
 & + \varepsilon \left(\frac{2a_3}{p} - C_1 k^{1-m} \right) \|u\|_p^p.
 \end{aligned} \tag{2.29}$$

At this point, we choose k large enough so that (2.29) becomes

$$\begin{aligned}
 L'(t) \geq & \left[(1 - \sigma) - \frac{m - 1}{m} \varepsilon k \right] H^{-\sigma}(t) \|u_t\|_m^m \\
 & + \varepsilon \gamma [H(t) + \|u_t\|_2^2 + \|u\|_p^p + (g \circ \nabla u)(t)],
 \end{aligned} \tag{2.30}$$

where $\gamma > 0$ is the minimum of the coefficients of $H(t)$, $\|u_t\|_2^2$, $\|u\|_p^p$, and $(g \circ \nabla u)(t)$ in (2.29). Once k is fixed (hence γ also), we pick ε small enough so that

$$(1 - \sigma) - \varepsilon k(m - 1)/m \geq 0$$

and

$$L(0) = H^{1-\sigma}(0) + \varepsilon \int_{\Omega} u_0 u_1(x) dx > 0.$$

Therefore, (2.30) takes on the form

$$L'(t) \geq \varepsilon \gamma [H(t) + \|u_t\|_2^2 + \|u\|_p^p + (g \circ \nabla u)(t)]. \tag{2.31}$$

Consequently, we have

$$L(t) \geq L(0) > 0, \quad \forall t \geq 0.$$

We now estimate

$$\left| \int_{\Omega} uu_t dx \right| \leq \|u\|_2 \|u_t\|_2 \leq C \|u\|_p \|u_t\|_2,$$

which implies

$$\left| \int_{\Omega} uu_t dx \right|^{1/(1-\sigma)} \leq C \|u\|_p^{1/(1-\sigma)} \|u_t\|_2^{1/(1-\sigma)}.$$

Again, Young’s inequality gives us

$$\left| \int_{\Omega} uu_t dx \right|^{1/(1-\sigma)} \leq C [\|u\|_p^{\mu/(1-\sigma)} + \|u_t\|_2^{\theta/(1-\sigma)}], \tag{2.32}$$

for $1/\mu + 1/\theta = 1$. To obtain $\mu/(1-\sigma) = 2/(1-2\sigma) \leq p$ by (2.20), we take $\theta = 2(1-\sigma)$. Therefore, (2.32) becomes

$$\left| \int_{\Omega} uu_t(x, t) dx \right|^{1/(1-\sigma)} \leq C [\|u\|_p^s + \|u_t\|_2^2],$$

where $s = 2/(1-2\sigma) \leq p$. Using Lemma 2.4, we obtain

$$\left| \int_{\Omega} uu_t dx \right|^{1/(1-\sigma)} \leq C [H(t) + \|u\|_p^p + \|u_t\|_2^2 + (g \circ \nabla u)(t)], \quad \forall t \geq 0. \tag{2.33}$$

Therefore, we have

$$\begin{aligned} L^{1/(1-\sigma)}(t) &= \left(H^{1-\sigma}(t) + \varepsilon \int_{\Omega} uu_t dx \right)^{1/(1-\sigma)} \\ &\leq 2^{1/(1-\sigma)} \left(H(t) + \left| \int_{\Omega} uu_t dx \right|^{1/(1-\sigma)} \right) \\ &\leq C [H(t) + \|u\|_p^p + \|u_t\|_2^2 + (g \circ \nabla u)(t)], \quad \forall t \geq 0. \end{aligned} \tag{2.34}$$

Combining (2.31) and (2.34), we arrive at

$$L'(t) \geq \Gamma L^{1/(1-\sigma)}(t), \quad \forall t \geq 0, \tag{2.35}$$

where Γ is a positive constant depending only on $\varepsilon\gamma$ and C . A simple integration of (2.35) over $(0, t)$ then yields

$$L^{\sigma/(1-\sigma)}(t) \geq \frac{1}{L^{-\sigma/(1-\sigma)}(0) - \Gamma t \sigma / (1 - \sigma)}. \tag{2.36}$$

Therefore, (2.36) shows that $L(t)$ blows up in time

$$T^* \leq \frac{1 - \sigma}{\Gamma \sigma [L(0)]^{\sigma/(1-\sigma)}}. \tag{2.37}$$

This completes the proof. \square

Remark 2.1. By following the steps of the proof of Theorem 2.5 closely, one can easily see that the blow-up result holds even for $m = 1$ (damping caused only by viscosity). A small modification is needed in the proof.

Remark 2.2. The third inequality in (1.5) shows that there is a strong relation between the nonlinearity in the source and the damping caused by the viscosity. More precisely, the larger p is, the closer $\int_0^\infty g(s) ds$ can be to 1.

Remark 2.3. The estimate (2.37) shows that the larger $L(0)$ is, the quicker the blow-up takes place.

Acknowledgment

The author expresses his sincere thanks to KFUPM for its support. This work has been funded by KFUPM under Project # MS/VISCO ELASTIC/270.

References

- [1] J. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolutions equations, *Quart. J. Math. Oxford* 28 (1977) 473–486.
- [2] S. Berrimi, S.A. Messaoudi, Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping, *Electron. J. Differential Equations* 2004 (2004) 1–10.
- [3] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J. Ferreira, Existence and uniform decay for a non-linear viscoelastic equation with strong damping, *Math. Methods Appl. Sci.* 24 (2001) 1043–1053.
- [4] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.A. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping, *Electron. J. Differential Equations* 2002 (2002) 1–14.
- [5] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.S. Prates Filho, J.A. Soriano, Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping, *Differential Integral Equations* 14 (2001) 85–116.
- [6] M.M. Cavalcanti, H.P. Oquendo, Frictional versus viscoelastic damping in a semilinear wave equation, *SIAM J. Control Optim.* 42 (2003) 1310–1324.

- [7] V. Georgiev, G. Todorova, Existence of solutions of the wave equation with nonlinear damping and source terms, *J. Differential Equations* 109 (1994) 295–308.
- [8] A. Haraux, E. Zuazua, Decay estimates for some semilinear damped hyperbolic problems, *Arch. Ration. Mech. Anal.* 150 (1988) 191–206.
- [9] V.K. Kalantarov, O.A. Ladyzhenskaya, The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type, *J. Soviet Math.* 10 (1978) 53–70.
- [10] M. Kopackova, Remarks on bounded solutions of a semilinear dissipative hyperbolic equation, *Comment. Math. Univ. Carolin.* 30 (1989) 713–719.
- [11] H.A. Levine, Instability and nonexistence of global solutions of nonlinear wave equation of the form $Pu_{tt} = Au + F(u)$, *Trans. Amer. Math. Soc.* 192 (1974) 1–21.
- [12] H.A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM J. Math. Anal.* 5 (1974) 138–146.
- [13] H.A. Levine, J. Serrin, Global nonexistence theorems for quasilinear evolution equation with dissipation, *Arch. Ration. Mech. Anal.* 137 (1997) 341–361.
- [14] H.A. Levine, S. Ro Park, Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation, *J. Math. Anal. Appl.* 228 (1998) 181–205.
- [15] S.A. Messaoudi, Blow up in a nonlinearly damped wave equation, *Math. Nachr.* 231 (2001) 1–7.
- [16] S.A. Messaoudi, N.-E. Tatar, Global existence and asymptotic behavior for a nonlinear viscoelastic problem, *Math. Meth. Sci. Res. J.* 7 (2003) 136–149.
- [17] S.A. Messaoudi, Blow up and global existence in a nonlinear viscoelastic equation, *Math. Nachr.* 260 (2003) 58–66.
- [18] S.A. Messaoudi, B. Said-Houari, Blow up of solutions of a class of wave equations with nonlinear damping and source terms, *Math. Methods Appl. Sci.* 27 (2004) 1687–1696.
- [19] E. Vitillaro, Global nonexistence theorems for a class of evolution equations with dissipation, *Arch. Ration. Mech. Anal.* 149 (1999) 155–182.
- [20] Zhijian Yang, Existence and asymptotic behavior of solutions for a class of quasi-linear evolution equations with non-linear damping and source terms, *Math. Methods Appl. Sci.* 25 (2002) 795–814.