BLOW-UP OF SOLUTIONS OF A SEMILINEAR HEAT EQUATION WITH A MEMORY TERM

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We consider an initial boundary value problem related to the equation $u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = |u|^{p-2}u$ and prove, under suitable conditions on g and p, a blow-up result for certain solutions with positive initial energy.

1. Introduction

In this paper, we are concerned with the finite-time blow-up of solutions for the initial boundary value problem

$$\begin{gathered} u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = |u|^{p-2}u, \quad x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1 function, $p > 2$, and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$.

This equation arises from a variety of mathematical models in engineering and physical sciences. For example, in the study of heat conduction in materials with memory, the classical Fourier's law of heat flux is replaced by the following form:

$$q = -d\nabla u - \int_{-\infty}^t \nabla[k(x,t)u(x,\tau)]d\tau, \tag{1.2}$$

where u is the temperature, d is the diffusion coefficient, and the integral term represents the memory effect in the material. The study of this type of equations has drawn a considerable attention, see [3, 4, 10, 12, 13]. From a mathematical point of view, one would expect the integral term to be dominated by the leading term in the equation. Therefore, the theory of parabolic equations applies to this type of equations.

In the absence of the memory term ($g = 0$), problem (1.1) has been studied by various authors and several results concerning global and nonglobal existence have been established. For instance, in the early 1970s, Levine [6] introduced the concavity method and

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showed that solutions with negative energy blow up in finite time. Later, this method was improved by Kalantarov and Ladyzhenskaya [5] to accommodate more general situations. Ball [2] also studied (1.1) with $f(u,\nabla u)$ instead of $|u|^{p-2}u$ and established a nonglobal existence result in bounded domains. This result had been extended to unbounded domains by Alfonsi and Weissler [1].

For the quasilinear case, Junning in [14] studied

$$\begin{gathered} u_t - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) = f(u), \quad x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.3}$$

and established a global existence result. He also proved a nonglobal existence result under the condition

$$\frac{1}{m}\int_\Omega |\nabla u_0(x)|^m dx - \int_\Omega F(u_0(x))\,dx \leq -\frac{4(m-1)}{mT(m-2)^2}\int_\Omega u_0^2(x)dx, \tag{1.4}$$

where $F(u) = \int_0^u f(s)ds$. More precisely he showed that if there exists $T > 0$, for which (1.2) holds, then the solution blows up in a time less than T. This type of results have been extensively generalized and improved by Levine, Park, and Serrin in [7], where the authors proved some global, as well as nonglobal, existence theorems. Their result, when applied to problem (1.3), requires that

$$\frac{1}{m}\int_\Omega |\nabla u_0(x)|^m dx - \int_\Omega F(u_0(x))\,dx < 0. \tag{1.5}$$

We note that the inequality (1.5) implies (1.4). In a note, Messaoudi [8] extended the blow-up result to a solution with an initial datum satisfying

$$\frac{1}{m}\int_\Omega |\nabla u_0(x)|^m dx - \int_\Omega F(u_0(x))\,dx \leq 0. \tag{1.6}$$

In the present work, we consider (1.1) and show that, for suitable conditions on p and g, the blow-up can be obtained even for some solutions with positive initial energy. The present paper improves the one in [8] as it is only a special case.

2. Blow-up

In order to state and prove our result, we introduce the "modified" energy functional

$$E(t) = \frac{1}{2}(g \diamond \nabla u)(t) + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 - \frac{1}{p}\|u(t)\|_p^p, \tag{2.1}$$

where

$$(g \diamond v)(t) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|_2^2 d\tau. \tag{2.2}$$

For the relaxation function g and the number p, we assume that

$$g(s) \geq 0, \qquad g'(s) \leq 0, \qquad 1 - \int_0^\infty g(s)ds = l > 0, \tag{2.3}$$

$$2 < p \leq \frac{2(n-1)}{n-2}, \quad n > 2, \; p > 2, \; n = 1,2. \tag{2.4}$$

We also set

$$\alpha = B^{-p/(p-2)}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)\alpha^2, \tag{2.5}$$

where $B = C_*/l$ for C_* the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$.

By multiplying the equation in (1.1) by u_t and integrating over Ω, we get, after some manipulations, see [9],

$$\frac{d}{dt}E(t) = -\left(\frac{1}{2}g(t)||\nabla u(t)||_2^2 - \frac{1}{2}(g' \diamond \nabla u)(t) + \int_\Omega |u_t|^2 u_t dx\right) \leq 0, \tag{2.6}$$

for regular solutions. The same result can be established, for almost every t, by a simple density argument.

Similar to [11], we give a definition for a strong solution of (1.1).

Definition 1. A strong solution of (1.1) is a function $u \in C([0,T);H_0^1(\Omega)) \cap C^1([0,T);L^2(\Omega))$, satisfying (2.6) and

$$\int_0^t \int_\Omega \left(\nabla u \cdot \nabla\phi - \int_0^s \nabla u(\tau) \cdot \nabla\phi(s)d\tau + u_t\phi - |u|^{p-2}u\phi\right)dx\,ds = 0, \tag{2.7}$$

for all t in $[0,T)$ and all ϕ in $C([0,T),H_0^1(\Omega))$.

Remark 2.1. Condition (2.4) is needed so that $|u|^{p-2}u \in L^2(\Omega)$; hence $\int_\Omega |u|^{p-2}u\phi\,dx$ makes sense. The condition $1 - \int_0^\infty g(s)ds = l > 0$ is necessary to guarantee the parabolicity of system (1.1).

LEMMA 2.2. *Let u be a strong solution of (1.1) with initial data satisfying*

$$E(0) < E_1, \qquad ||\nabla u_0||_2 > \alpha. \tag{2.8}$$

Then there exists a constant $\beta > \alpha$ such that

$$\left[\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\right]^{1/2} \geq \beta, \tag{2.9}$$

$$\|u\|_p \geq B\beta \quad \forall t \in [0,T). \tag{2.10}$$

Proof. We first note that, by (2.1) and the Sobolev embedding, we have

$$\begin{aligned} E(t) &= \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{p}\|u\|_p^p \\ &\geq \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{p}B^p l^p\|\nabla u\|_2^p \\ &\geq \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) \\ &\quad - \frac{B^p}{p}\left[\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)\right]^{p/2} \\ &= \frac{1}{2}\zeta^2 - \frac{B^p}{p}\zeta^p = h(\zeta), \end{aligned} \tag{2.11}$$

where

$$\zeta = \left[\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)\right]^{1/2}. \tag{2.12}$$

It is easy to verify that h is increasing for $0<\zeta<\alpha$, decreasing for $\zeta>\alpha$, $h(\zeta)\to-\infty$ as $\zeta\to+\infty$, and

$$h(\alpha) = \left(\frac{1}{2}-\frac{1}{p}\right)B^{-2p/(p-2)} = E_1, \tag{2.13}$$

where α is given in (2.8). Therefore, since $E(0)<E_1$, there exists $\beta>\alpha$ such that $h(\beta)=E(0)$.

By using (2.11) we have

$$h(||\nabla u_0||_2) \leq E(0) = g(\beta), \tag{2.14}$$

which implies that $\|\nabla u_0\|_2 \geq \beta$.

Now to establish (2.9), we suppose by contradiction that

$$\left[\left(1-\int_0^{t_0} g(s)ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)(t_0)\right]^{1/2} < \beta, \tag{2.15}$$

for some $t_0>0$ and, by the continuity of

$$\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)(t), \tag{2.16}$$

we can choose t_0 such that

$$\left[\left(1-\int_0^{t_0} g(s)ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)(t_0)\right]^{1/2} > \alpha. \tag{2.17}$$

Again the use of (2.11) leads to

$$E(t_0) \geq h\left(\left[\left(1-\int_0^{t_0} g(s)ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t_0)\right]^{1/2}\right) > h(\beta) = E(0). \tag{2.18}$$

This is impossible since $E(t) \leq E(0)$, for all $t \in [0,T)$. Hence (2.9) is established.

To prove (2.10), we exploit (2.1) and (2.6) to obtain

$$\frac{1}{2}\left[\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right] \leq E(0)+\frac{1}{p}\|u\|_p^p. \tag{2.19}$$

Consequently

$$\begin{aligned}\frac{1}{p}\|u\|_p^p &\geq \frac{1}{2}\left[\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right]-E(0)\\ &\geq \frac{1}{2}\beta^2-E(0)\\ &\geq \frac{1}{2}\beta^2-h(\beta)=\frac{B^p}{p}\beta^p.\end{aligned} \tag{2.20}$$

Therefore (2.20) yields the desired result. The proof is completed. □

THEOREM 2.3. *Assume that (2.3) and (2.4) hold. Given $u_0 \in H_0^1(\Omega)$ satisfying*

$$\|\nabla u_0\|_2 > \alpha, \qquad E(0) < E_1, \tag{2.21}$$

if

$$\int_0^\infty g(s)ds < \frac{1-c_0}{1-(3/4)c_0}, \quad c_0 = \frac{2+(p-2)(\alpha/\beta)^p}{p} < 1, \tag{2.22}$$

then any strong solution of (1.1) blows up in finite time.

Proof. We define

$$L(t) = \frac{1}{2}\int_\Omega u^2(x,t)dx \tag{2.23}$$

and differentiate L to get

$$\begin{aligned}L'(t) &= \int_\Omega uu_t(x,t)dx\\ &= \int_\Omega u\Delta u\,dx-\int_\Omega u(x,t)\int_0^t g(t-s)\Delta u(x,s)ds\,dx+\int_\Omega |u|^p dx\\ &= -\int_\Omega |\nabla u|^2dx+\int_\Omega\int_0^t g(t-s)\nabla u(x,t)\cdot\nabla u(x,s)ds\,dx+\int_\Omega |u|^p dx\\ &\geq -\int_\Omega |\nabla u|^2dx+\int_0^t g(t-s)\|\nabla u(t)\|_2^2 d\tau+\int_\Omega |u|^p dx\\ &\quad -\int_0^t g(t-s)\int_\Omega |\nabla u(t)\cdot[\nabla u(s)-\nabla u(t)]|\,dx\,d\tau.\end{aligned} \tag{2.24}$$

By using Schwarz inequality, (2.24) takes the form

$$\begin{aligned} L'(t) \geq & \int_\Omega |u|^p dx - \left(1 - \int_0^t g(s)ds\right) \|\nabla u(t)\|_2^2 \\ & - \int_0^t g(t-\tau)\|\nabla u(t)\|_2 \|\nabla u(\tau) - \nabla u(t)\|_2 d\tau. \end{aligned} \tag{2.25}$$

By applying Young's inequality to the last term of (2.25), we arrive at

$$L'(t) \geq \int_\Omega |u|^p dx - \left[1 - \frac{3}{4}\int_0^t g(s)ds\right] \|\nabla u(t)\|_2^2 - (g \diamond \nabla u)(t). \tag{2.26}$$

We then substitute for $\|\nabla u(t)\|_2^2$ from (2.1); hence (2.26) becomes

$$\begin{aligned} L'(t) \geq & \int_\Omega |u|^p dx + 2\frac{[1-(3/4)\int_0^t g(s)ds]}{(1-\int_0^t g(s)ds)} H(t) - 2\frac{[1-(3/4)\int_0^t g(s)ds]}{(1-\int_0^t g(s)ds)} E_1 \\ & + \left(\frac{1-(3/4)\int_0^t g(s)ds}{(1-\int_0^t g(s)ds)} - 1\right)(g \diamond \nabla u)(t) \\ & - \frac{2}{p}\frac{1-(3/4)\int_0^t g(s)ds}{(1-\int_0^t g(s)ds)} \int_\Omega |u|^p dx. \end{aligned} \tag{2.27}$$

By using (2.5) and (2.9), the estimate (2.27) takes the form

$$\begin{aligned} L'(t) \geq & \, 2\frac{[1-(3/4)\int_0^t g(s)ds]}{(1-\int_0^t g(s)ds)} H(t) + \left(\frac{[1-(3/4)\int_0^t g(s)ds]}{(1-\int_0^t g(s)ds)} - 1\right)(g \diamond \nabla u)(t) \\ & + \left[1 - \left(\frac{2}{p} + \frac{p-2}{p}\left(\frac{\alpha}{\beta}\right)^p\right)\frac{1-(3/4)\int_0^t g(s)ds}{(1-\int_0^t g(s)ds)}\right] \int_\Omega |u|^p dx \\ \geq & \, \gamma \int_\Omega |u|^p dx, \end{aligned} \tag{2.28}$$

where

$$\gamma = 1 - \left(\frac{2}{p} + \frac{p-2}{p}\left(\frac{\alpha}{\beta}\right)^p\right)\frac{1-(3/4)\int_0^\infty g(s)ds}{(1-\int_0^\infty g(s)ds)} > 0 \tag{2.29}$$

because of (2.22). Next we have, by the embedding of the L^q spaces,

$$L^{p/2}(t) \leq C\|u\|_p^p. \tag{2.30}$$

By combining (2.28) and (2.30) we get

$$L'(t) \geq \Gamma L^{p/2}(t). \tag{2.31}$$

A direct integration of (2.31) then yields

$$L^{p/2-1}(t) \geq \frac{1}{L^{1-p/2}(0) - \Gamma t}. \tag{2.32}$$

Therefore L blows up in a time $t^* \leq 1/\Gamma L^{(p/2)-1}(0)$. □

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