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# Decay of Solutions of a Nonlinear Hyperbolic System Describing Heat Propagation by Second Sound

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In this work we establish a decay result for the solutions of a nonlinear hyperbolic system describing heat propagation, where the heat flux is given by Cattaneo's law.

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## 1 INTRODUCTION

In the absence of deformation and external sources, the evolution of the heat flux and the absolute temperature is given by the system

$$\begin{aligned} q + \kappa(\theta)\theta_x &= 0 \\ q_x + c(\theta)\theta_t &= 0, \end{aligned} \tag{1.1}$$

where $\kappa$ and $c$ are strictly positive functions characterizing the material in consideration. In the case where $c$ and $\kappa$ are independent of $\theta$, we get

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the familiar linear heat equation

$$\theta_t = k\theta_{xx}, \quad k = \frac{\kappa}{c}.$$

This equation provides a useful description of heat conduction under a large range of conditions and predicts an infinite speed of propagation; that is, any thermal disturbance at one point has an instantaneous effect elsewhere in the body. This is not always the case. In fact, experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox (infinite speed propagation) and disturbances which are almost entirely thermal may propagate in a finite speed. This phenomenon in dielectric crystals is called second sound.

These observations go back to 1948, when Cattaneo [1] proposed, instead of Fourier's law, a new constitutive relation

$$\tau(\theta)q_t + q = -\kappa(\theta)\,\theta_x, \tag{1.2}$$

where $\tau$ and $\kappa$ are strictly positive functions depending on the absolute temperature. Coleman *et al.* [2] showed in 1982 that, if (1.2) is adopted then compatibility with thermodynamics requires that the internal energy be given by

$$e = \tilde{e}(\theta, q) = a(\theta) + b(\theta)\,q^2, \tag{1.3}$$

where $b$ is a function determined by $\tau$ and $\kappa$. In particular $b(\theta) > 0$. In this case the system governing the evolution of $\theta$ and $q$ takes the form

$$\begin{aligned} q_x + (a'(\theta) + b'(\theta)q^2)\theta_t + 2b(\theta)\,qq_t &= 0 \\ \tau(\theta)q_t + q + \kappa(\theta)\,\theta_x &= 0. \end{aligned} \tag{1.4}$$

Global existence and decay of classical solutions to the Cauchy problem, as well as to some initial boundary value problems, have been established by Coleman *et al.* [3]. They also showed that $(\theta, q)$ tends to the equilibrium state, however, no rate of decay has been discussed.

Concerning the formation of singularities, Messaoudi [4] studied the following system

$$\begin{aligned} \tau(\theta)q_t + q + \kappa(\theta)\,\theta_x &= 0 \\ c(\theta)\theta_t + q_x &= 0 \end{aligned}$$

and showed, under the same restrictions on $\tau$, $c$ and $\kappa$, that classical solutions to the Cauchy problem break down in finite time if the initial data are chosen small in the $L^\infty$ norm with large enough derivatives. This result has been improved later and established by the author [5] for an equivalent system of the form

$$\begin{aligned} \sigma(e,q)q_t + \mu(e,q)q &= -e_x + \lambda(e,q)\,qq_x \\ e_t &= -q_x, \end{aligned} \tag{1.5}$$

where $\sigma, \mu$ satisfy

$$\sigma(\xi,\eta) \geq \underline{\sigma} > 0, \quad \mu(\xi,\eta) \geq \underline{\mu} > 0, \quad \forall(\xi,\eta) \in \mathbb{R}^2. \tag{1.6}$$

For the derivation of (1.5) from (1.4) and the proof of the global existence, we refer the reader to [5,6].

In this work we consider (1.5) together with the initial and boundary conditions

$$\begin{aligned} &e(x,0) = e_0(x), \quad q(x,0) = q_0(x), \quad x \in I = (0,1) \\ &q(0,t) = q(1,t) = 0, \quad t \geq 0 \end{aligned}$$

and show that global classical solutions decay exponentially if the initial data are sufficiently small.

## 2 EXPONENTIAL DECAY

In this section, we state and prove our main result. For this purpose we set

$$\hat{e} = e - e_1, \quad e_1 = \int_0^1 e_0(x)\,dx.$$

It is clear that $(\hat{e}, q)$ satisfies the following problem

$$\hat{\sigma}(\hat{e}, q)q_t + \hat{\mu}(\hat{e}, q)q = -\hat{e}_x + \hat{\lambda}(\hat{e}, q)\, qq_x \tag{2.1}$$

$$\hat{e}_t = -q_x, \tag{2.2}$$

$$\hat{e}(x, 0) = \hat{e}_0(x), \quad q(x, 0) = q_0(x), \quad x \in I \tag{2.3}$$

$$q(0, t) = q(1, t) = 0, \quad t \geq 0, \tag{2.4}$$

where

$$\hat{\sigma}(\hat{e}, q) = \sigma(\hat{e} + e_1, q), \hat{\mu}(\hat{e}, q) = \mu(\hat{e} + e_1, q), \hat{\lambda}(\hat{e}, q) = \lambda(\hat{e} + e_1, q).$$

*Remark 2.1* In the sequel we go back to the notation $e$ instead of $\hat{e}$.

*Remark 2.2* By using (2.2), we easily see that $\hat{e}$ satisfies Poincare's inequality.

THEOREM *Assume that $\sigma, \mu, \lambda$ are $C^2$ functions satisfying* (1.6). *Then there exists a small positive constant $\delta$ such that for any $e_0$ in $H^2(I)$ and $q_0$ in $H^2(I) \cap H_0^1(I)$ satisfying*

$$\|e_0\|_2^2 + \|q_0\|_2^2 < \delta^2, \tag{2.5}$$

*the solution of* (2.1)–(2.4) *decays exponentially as $t \to +\infty$.*

In order to carry out the proof, we consider another problem which agrees with (3.1)–(3.4) when $(e, q)$ are close enough to the equilibrium state $(0, 0)$. For this purpose, we introduce the functions $A, B, C$ satisfying the following hypotheses

(h1) $A, B, C \in C_b^2(\mathbb{R}^2)$
(h2) $A(\xi, \eta) = \hat{\sigma}(\xi, \eta),\ B(\xi, \eta) = \hat{\mu}(\xi, \eta),\ C(\xi, \eta) = \hat{\lambda}(\xi, \eta),\ \forall(\xi, \eta) \in \mathcal{V}$ a neighborhood of $(0, 0)$
(h3) $A(\xi, \eta) \geq \underline{A} > 0,\ B(\xi, \eta) \geq \underline{B} > 0.$

Here $C_b^2$ denotes the space of continuous and bounded functions, as well as, their first and second order derivatives. We note that functions with these properties can be constructed by virtue of (1.6). Therefore, instead of (3.1)–(3.4) we consider the following problem

$$A(e, q)q_t + B(e, q)q = -e_x + qC(e, q)\, q_x \tag{2.6}$$

$$e_t = -q_x \quad x \in I, \quad t \geq 0 \tag{2.7}$$

$$e(x,0) = e_0(x), \quad q(x,0) = q_0(x), \quad x \in I. \tag{2.8}$$

$$q(0,t) = q(1,t) = 0, \quad t \geq 0, \tag{2.9}$$

*Remark 2.3* Any solution $(e,q)$ to (2.6)–(2.9) satisfying $(e,q) \in \mathcal{V}$ is also a solution to (2.1)–(2.4) by virtue of (h2).

We also set

$$\mathcal{E}(t) := \int_0^1 (e^2 + e_t^2 + e_x^2 + e_{xt}^2 + e_{xx}^2 + e_{tt}^2 + q^2 + q_t^2 + q_x^2 + q_{xt}^2 + q_{xx}^2 + q_{tt}^2](x,t)\,dx \tag{2.10}$$

$$\Lambda(t) := \int_0^1 [e^2 + e_t^2 + e_x^2 + e_{xt}^2 + e_{tt}^2](x,t)\,dx + \int_0^1 A(e,q)[q^2 + q_t^2 + q_x^2 + q_{xt}^2 + q_{tt}^2)(x,t)\,dx \tag{2.11}$$

$$\alpha(t) := \sup_{0 \leq x \leq 1} [(|e| + |e_x| + |e_t| + |q| + |q_t| + |q_x|)(x,t)] \tag{2.12}$$

*Proof* We multiply (2.6) by $q$ and (2.7) by $e$, integrate over $I$, use integration by parts, and add equalities, to obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [Aq^2 + e^2](x,t)dx \leq -\int_0^1 Bq^2(x,t)dx + \Gamma\alpha(t)\,\mathcal{E}(t), \tag{2.13}$$

where $\Gamma$ denotes a positive (possibly large) generic constant independent of $e, q, t$.

To get the next estimates, we differentiate (2.6), (2.7) with respect to

$$Aq_{tt} + A_t q_t + Bq_t + B_t q = -e_{xt} + C_t q q_x + C q q_{xt} + C q_t q_x \tag{2.14}$$

$$e_{tt} = -q_{xt} \tag{2.15}$$

and multiply (2.14) by $q_t$ and (2.15) by $e_t$. Similar computations as above then yield

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[Aq_t^2 + e_t^2\right](x,t)\,dx \le -\int_0^1 Bq_t^2(x,t)\,dx + \Gamma(\alpha(t)+\alpha^2(t))\mathcal{E}(t). \tag{2.16}$$

To establish bounds on terms involving $e_{tt}$ and $q_{tt}$, we introduce the difference operator as follows: for $h > 0$, we set

$$\Delta_h W(x,t) = W(x,t+h) - W(x,t), \quad x \in \mathbb{R}, \quad t \ge 0 \tag{2.17}$$

We apply the above operator to Eqs. (2.14), (2.15), multiply the resulting equalities by $\Delta_h q_t$ and $\Delta_h e_t$ respectively, integrate over $I$, and add the inequalities. After a number of integrations, using integration by parts, we divide by $h^2$ and let $h$ go to zero. Thus we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [Aq_{tt}^2 + e_{tt}^2](x,t)\,dx \le -\int_0^1 Bq_{tt}^2(x,t)\,dx + \Gamma(\alpha(t)+\alpha^2(t)+\alpha^3(t))\mathcal{E}(t). \tag{2.18}$$

For additional estimates, we differentiate (2.6), (2.7) with respect to $x$ to get

$$Aq_{xt} + A_xq_t + Bq_x + B_xq = -e_{xx} + C_xqq_x + Cqq_{xx} + Cq_x^2 \tag{2.19}$$

$$e_{xt} = -q_{xx}. \tag{2.20}$$

We then multiply (2.19) by $q_x$ and (2.20) by $e_x$ to obtain, by similar calculations,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[Aq_x^2 + e_x^2\right](x,t)\,dx \le -\int_0^1 Bq_x^2(x,t)dx + \Gamma(\alpha(t)+\alpha^2(t))\mathcal{E}(t). \tag{2.21}$$

Again we apply the operator (2.17) to Eqs. (2.19), (2.20), multiply the resulting equalities by $\Delta_h q_x$ and $\Delta_h e_x$ respectively, integrate over $I$, and add the inequalities. After a number of integrations, using integration by parts, we divide by $h^2$ and let $h$ go to zero to arrive at

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[Aq_{xt}^2 + e_{xt}^2\right](x,t)\,dx \le -\int_0^1 Bq_{xt}^2(x,t)\,dx + \Gamma(\alpha(t)+\alpha^2(t)+\alpha^3(t))\mathcal{E}(t). \qquad (2.22)$$

By combining (2.13), (2.16), (2.18), (2.21) and (2.22), we get

$$\Lambda'(t) \le -\int_0^1 B\big(q^2 + q_t^2 + q_x^2 + q_{tt}^2 + q_{xt}^2\big)(x,t)\,dx + \Gamma(\alpha(t)+\alpha^2(t)+\alpha^3(t))\mathcal{E}(t). \qquad (2.23)$$

Next we show that, for $(e,q)$, $\Lambda$ is equivalent to $\mathcal{E}$. For this purpose we use Eqs. (2.6), (2.7), (2.14), (2.15), (2.19), (2.20) and the hypotheses (h1)–(h3). Thus, we obtain

$$c_1(1+\alpha^2(t)+\alpha^4(t))\mathcal{E}(t) \le \Lambda(t) \le c_2\mathcal{E}(t) \qquad (2.24)$$

where $c_1, c_2$ are constants. A combination of (2.23) and (2.24) then yields

$$\Lambda'(t) \le -\int_0^1 B\big(q^2 + q_t^2 + q_x^2 + q_{tt}^2 + q_{xt}^2\big)(x,t)\,dx + \Gamma\alpha(t)\Lambda(t). \qquad (2.25)$$

Next we exploit (2.6), (2.7), (2.14), (2.15), (2.19), (2.20) for further estimates

$$\int_0^1 \big(e_t^2 + e_{tt}^2\big)(x,t)\,dx = \int_0^1 \big(q_x^2 + q_{xt}^2\big)(x,t)\,dx \qquad (2.26)$$

and

$$\int_0^1 \big(e_x^2 + e_{xt}^2\big)(x,t)\,dx \le c\int_0^1 \big(q^2 + q_t^2 + q_{tt}^2\big)(x,t)\,dx + \Gamma\alpha(t)\Lambda(t). \qquad (2.27)$$

We then use Poincare's inequality and (2.6) to obtain

$$\int_0^1 e^2(x,t)\,dx \le c\int_0^1 \left(q^2+q_t^2\right)(x,t)\,dx + \Gamma\alpha(t)\Lambda(t). \tag{2.28}$$

By combining (2.25)–(2.28) and using (h3) we conclude

$$\Lambda'(t) \le -a\Lambda(t) + \Gamma\alpha(t)\Lambda(t). \tag{2.29}$$

where $a$ is a constant depending only on the upper and lower bounds of $A, B, C$. We also note that, by the standard Sobolev embedding inequalities we have $\alpha(t) \le \sqrt{2\mathcal{E}(t)}$. So by choosing $\delta$ in (2.5) so small that $\Gamma\alpha(0) < a/2$ and $(e,q) \in \mathcal{V}$, the relation (2.29) yields

$$\Lambda'(t) < -\frac{a}{2}\Lambda(t), \quad \forall t \in [0,\varepsilon) \tag{2.30}$$

Direct integration then leads to

$$\Lambda(t) \le \Lambda(0)e^{-at/2}, \qquad \forall t \in [0,\varepsilon). \tag{2.31}$$

Since $\Lambda(t) \le \Lambda(0)$ we extend (2.31) beyond $\varepsilon$. By repeating the same procedure and using the continuity of $\Lambda$, (2.31) is established for all $t \ge 0$. This completes the proof. ■

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