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A Decay Result for a Quasilinear Parabolic System

Said Berrimi and Salim A. Messaoudi

Dedicated to Pr. Haim. Brezis on the occasion of his 60th birthday

Abstract. In this paper we consider a quasilinear parabolic system of the form

$$A(t)\,|u_t|^{m-2}\,u_t - \Delta u = u\,|u|^{p-2}\,,$$

$m \geq 2$, $p > 2$, in a bounded domain associated with initial and Dirichlet boundary conditions.We show that, for suitable initial datum, the energy of the solution decays " in time" exponentially if $m = 2$ whereas the decay is of a polynomial order if $m > 2$.

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1. Introduction

Research of global existence and finite time blow-up of solutions for the initial boundary value problem

$$\begin{aligned} u_t - div(|\nabla u|^{\alpha-2}\nabla u) + f(u) = 0, &\qquad x \in \Omega, \quad t > 0 \\ u(x,t) = 0, &\qquad x \in \partial\Omega, \quad t \geq 0 \\ u(x,0) = u_0(x), &\qquad x \in \Omega, \end{aligned} \tag{1}$$

where $\alpha \geq 2$ and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$, has attracted a great deal of people. The obtained results show that global existence and nonexistence depend roughly on α, the degree of nonlinearity in f, the dimension n, and the size of the initial datum. In the early 70's, Levine [8] introduced the concavity method and showed that solutions with negative energy blow-up in finite time. Later, this method had been improved by Kalantarov and Ladyzhenskaya [7] to accommodate more situations. Ball [2] also studied (1) with f depending on u as well as on ∇u and established a nonglobal existence result in bounded domains. This result was generalized to unbounded domains by Alfonsi and Weissler [1].

For the case $\alpha > 2$, Junning [6] studied (1) with f depending also on u and ∇u. He proved a nonglobal existence result under the condition

$$\frac{1}{m}\int_\Omega |\nabla u_0(x)|^m \, dx - \int_\Omega F(u_0(x))dx \leq -\frac{4(m-1)}{mT(m-2)^2}\int_\Omega u_0^2(x)dx, \tag{2}$$

where $F(u) = \int_0^u f(s)ds$. This type of results have been extensively generalized and improved by Levine, Park, and Serrin in a paper [9], where the authors proved some global, as well as nonglobal, existence theorems. Their result, when applied to problem (1), requires that

$$\frac{1}{m}\int_\Omega |\nabla u_0(x)|^m dx - \int_\Omega F(u_0(x))dx < 0. \tag{3}$$

We note that the inequality (3) implies (2). In 1999, Erdem [4] discussed the initial Dirichlet-type boundary problem for

$$u_t - \sum_{i=1}^n \frac{\partial}{\partial x_i}((d + |\nabla u|^{m-2})\frac{\partial u}{\partial x_i}) + g(u, \nabla u) = f(u), \quad x \in \Omega, \quad t > 0$$

and established a blow-up result. Messaoudi [10] showed that the blow-up result can also be obtained for solutions satisfying

$$\frac{1}{m}\int_\Omega |\nabla u_0(x)|^m dx - \int_\Omega F(u_0(x))dx \leq 0.$$

On the other hand if f has at most a linear growth then we can find global solutions (see [5]).

Concerning the asymptotic behavior, Engler, Kawohl, and Luckhaus [3] considered problem (1) with $\alpha = 2$ and showed that for, $f(0) = 0$, $f'(u) \geq a > 0$, and sufficiently small initial datum u_0, the solution satisfies a gradient estimate of the type

$$||\nabla u||_p \leq Ce^{-\delta t}||\nabla u_0||_p.$$

For initial boundary problems to the quasilinear equation

$$u_t - div(\sigma(|\nabla u|^2)\nabla u) + f(u, \nabla u) = 0,$$

results concerning global existence and gradient estimates have been established, under certain geometric conditions on $\partial\Omega$, by Nakao and Ohara [12], [13] and Nakao and Chen [14].

Pucci and Serrin [15] discussed the following quasilinear parabolic system

$$A(t)|u_t|^{m-2}u_t = \Delta u - f(x, u),$$

for $m > 1$ and f satisfying $(f(x, u), u) \geq 0$. They established a global result of solutions and showed that these solutions tend to the rest state as $t \to \infty$, however no rate of decay has been given.

In this work we consider a similar problem of the form

$$\begin{aligned} A(t)\,|u_t|^{m-2}\,u_t - \Delta u &= |u|^{p-2}\,u, && x \in \partial\Omega, \quad t \in J \\ u(x,t) &= 0, && x \in \partial\Omega, \quad t \in J \\ u(x,0) &= u_0, && x \in \Omega, \end{aligned} \tag{4}$$

where $J = [0,\infty)$ and Ω is a bounded open subset of R^n. The values of u are taken in R^N, $N \geq 1$ and $A \in C(J;\, R^{N\times N})$. We assume that A is bounded and satisfies the condition

$$(A(t)v, v) \geq c_0\,|v|^2\,, \qquad \forall t \in J, \quad v \in R^N,$$

where $(.,.)$ is the inner product in R^N and $c_0 > 0$. We will show that, for small initial energy, the solution of (4) decays exponentially if $m = 2$ whereas the decay is of a polynomial order if $m > 2$. Our method of proof relies on the use of a lemma by Nakao [11].

2. Preliminaries

In order to state and prove our result, we introduce the following notation:

$$\begin{aligned} I(u(t)) &= I(t) = \|\nabla u(t)\|_2^2 - \|u(t)\|_p^p \\ E(u(t)) &= E(t) = \tfrac{1}{2}\,\|\nabla u(t)\|_2^2 - \tfrac{1}{p}\,\|u(t)\|_p^p \\ H &= \left\{ v \in \left(H_0^1\right)^N : I(v) > 0 \right\} \cup \{0\}. \end{aligned} \tag{5}$$

By multiplying the equation in (4) by u_t and integrating over Ω, using the boundary conditions, we get

$$\frac{d}{dt}E(t) = -\int_\Omega A(t)\,|u_t|^{m-2}\,u_t.u_t dx \leq 0, \tag{6}$$

for regular solutions. The same result is obtained for weak solutions by a simple density argument.

Next,we prove the invariance of the set H. For this aim we note that, by the embedding $H_0^1 \hookrightarrow L^q$, we have

$$\|u\|_q \leq C\,\|\nabla u\|_2\,, \tag{7}$$

for $2 \leq q \leq \frac{2n}{n-2}$ if $n \geq 3$, $q > 2$ if $n = 1,2$ where $C = C(n,q,\Omega)$ is the best constant.

Lemma 2.1. (Nakao[11]) *Let $\varphi(t)$ be a nonincreasing and nonnegative function defined on $[0,T]$, $T > 1$, satisfying*

$$\varphi^{1+r}(t) \leq k_0(\varphi(t) - \varphi(t+1)), \qquad t \in [0,T]\,,$$

for $k_0 > 1$ and $r \geq 0$. Then we have , for each $t \in [0,T]$,

$$\begin{aligned} \varphi(t) &\leq \varphi(0)e^{-k[t-1]^+}, && r = 0 \\ \varphi(t) &\leq \left\{\varphi(0)^{-r} + k_0 r\,[t-1]^+\right\}^{\frac{-1}{r}} && r > 0 \end{aligned}$$

where $[t-1]^+ = \max\{t-1, 0\}$ and $k = \ln(\frac{k_0}{k_0-1})$.

Lemma 2.2. *Suppose that*

$$\begin{array}{ll} 2 < p \leq \frac{2n}{n-2}, & n \geq 3 \\ p > 2, & n = 1, 2. \end{array} \tag{8}$$

If $u_0 \in H$, *and satisfying*

$$C^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} < 1 \tag{9}$$

then the solution $u(t) \in H$ *for each* $t \in [0, T)$.

Proof. Since $I(u_0) > 0$, then there exists (by continuity) $T_m < T$ such that

$$I(u(t)) \geq 0, \ \forall t \in [0, T_m];$$

this gives

$$E(t) = \left(\frac{p-2}{2p} \right) \|\nabla u(t)\|_2^2 + \frac{1}{p} I(t) \geq \left(\frac{p-2}{2p} \right) \|\nabla u(t)\|_2^2. \tag{10}$$

So,

$$\|\nabla u(t)\|_2^2 \leq \left(\frac{2p}{p-2} \right) E(t) \leq \left(\frac{2p}{p-2} \right) E(0), \ \forall t \in [0, T_m]. \tag{11}$$

We then use (7)–(9) and (11) to obtain, for each $t \in [0, T_m]$,

$$\begin{aligned} &\|u(t)\|_p^p \leq C^p \|\nabla u(t)\|_2^p = C^p \|\nabla u(t)\|_2^{p-2} \|\nabla u(t)\|_2^2 \\ &\leq C^p \left(\tfrac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} \|\nabla u(t)\|_2^2 < \|\nabla u(t)\|_2^2. \end{aligned} \tag{12}$$

Therefore, by virtue of (5) and (12), we obtain

$$I(t) = \|\nabla u(t)\|_2^2 - \|u(t)\|_p^p > 0. \tag{13}$$

This shows that $u(t) \in H$, for all $t \in [0, T_m]$. By repeating this procedure, and using the fact that

$$\lim_{t \to T_m} C^p \left(\frac{2p}{p-2} E(t) \right)^{\frac{p-2}{2}} \leq \beta < 1,$$

T_m is extended to T.

Lemma 2.3. *Suppose that* (8) *and* (9) *hold, then*

$$\eta \|\nabla u(t)\|_2^2 \leq I(t). \tag{14}$$

Proof. It suffices to rewrite (12) as:

$$\begin{aligned} \|u(t)\|_p^p &\leq C^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} \|\nabla u(t)\|_2^2 = (1 - \eta) \|\nabla u(t)\|_2^2 \\ &\leq \|\nabla u(t)\|_2^2 - \eta \|\nabla u(t)\|_2^2. \end{aligned} \tag{15}$$

Thus (14) follows for

$$\eta = 1 - C^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} > 0. \tag{16}$$

Theorem. *Suppose that* (8) *holds. Assume further that* $u_0 \in H$ *and satisfies* (9), *then the solution satisfies the following decay estimations:*

$$E(t) \leq E(0) e^{-[t-1]^+}, \qquad m = 2 \tag{17}$$

$$E(t) \leq \left\{ (E(0))^{-(\frac{m-2}{2})} + \frac{C_5}{c_0} \frac{m-2}{2} [t-1]^+ \right\}^{-(\frac{2}{m-2})}, \qquad m > 2. \tag{18}$$

Proof. We integrate (6) over $[t, t+1]$ to obtain

$$\begin{aligned} E(t) - E(t+1) &= \int_t^{t+1} \int_\Omega |u_t(s)|^{m-2} A(s) u_t.u_t dx ds \\ &\geq c_0 \int_t^{t+1} \int_\Omega |u_t(s)|^m dx ds = c_0 (F(t))^m, \end{aligned} \tag{19}$$

where

$$(F(t))^m = \int_t^{t+1} \|u_t(s)\|_m^m ds. \tag{20}$$

Now we multiply the equation in (4) by u and integrate over $\Omega \times [t, t+1]$ to arrive at

$$\int_t^{t+1} I(s) ds \leq \int_t^{t+1} \|A(s)\| \int_\Omega |u_t(s)|^{m-1} |u(s)| dx ds.$$

By the Cauchy-Schwarz inequality, we have the following

$$\begin{aligned} \int_t^{t+1} I(s) ds &\leq \int_t^{t+1} \|A(s)\| \|u_t(s)\|_m^{m-1} \|u(s)\|_m ds \\ &\leq A \int_t^{t+1} \|u_t(s)\|_m^{m-1} \|u(s)\|_m ds, \end{aligned} \tag{21}$$

where

$$A = \sup_J \|A(s)\| < \infty.$$

Exploiting (7) and (10), we obtain

$$\int_t^{t+1} I(s) ds \leq CA \left(\frac{2p}{p-2} \right)^{\frac{1}{2}} \left(\sup_{t \leq s \leq t+1} E^{\frac{1}{2}}(s) \right) \left(\int_t^{t+1} \|u_t(s)\|_m^{m-1} ds \right). \tag{22}$$

Now we use the fact that

$$\int_t^{t+1} \left(\int_\Omega |u_t(s)|^m dx \right)^{\frac{m-1}{m}} ds \leq \left(\int_t^{t+1} \int_\Omega |u_t(s)|^m dx ds \right)^{\frac{m-1}{m}} = (F(t))^{m-1} \tag{23}$$

to get

$$\int_t^{t+1} I(s)ds \le CA\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}\left(E^{\frac{1}{2}}(t)\right)(F(t))^{m-1}. \tag{24}$$

From (5) we have

$$E(t) = \left(\frac{p-2}{2p}\right)\|\nabla u(t)\|_2^2 + \frac{1}{p}I(t). \tag{25}$$

Integrating both sides of (25) over $[t, t+1]$ and using (14), one can write

$$\int_t^{t+1} E(s)ds \le \left(\frac{1}{p} + \frac{p-2}{2p\eta}\right)\int_t^{t+1} I(s)ds. \tag{26}$$

A combination of (24) and (26) leads to

$$\int_t^{t+1} E(s)ds \le CA\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}\left(\frac{1}{p} + \frac{p-2}{2p\eta}\right)\left(E^{\frac{1}{2}}(t)\right)(F(t))^{m-1}. \tag{27}$$

By using (6) again, we have

$$E(s) \ge E(t+1), \quad \forall s \le t+1;$$

hence

$$\int_t^{t+1} E(s)ds \ge E(t+1). \tag{28}$$

Inserting (28) in (19) and using (27), we easily have

$$\begin{aligned} E(t) &\le \int_t^{t+1} E(s)ds + \int_t^{t+1}\int_\Omega A(s)\,|u_t(s)|^{m-2}\,u_t(s).u_t(s)dxds \\ &\le CA\left(\frac{2p}{p-2}\right)^{\frac{1}{2}}\left(\frac{1}{p} + \frac{p-2}{2p\eta}\right)E^{\frac{1}{2}}(t)(F(t))^{m-1} \\ &\quad + \int_t^{t+1}\int_\Omega A(s)\,|u_t(s)|^m\,dxds \\ &\le C_1\left[E^{\frac{1}{2}}(t)(F(t))^{m-1} + (F(t))^m\right], \end{aligned} \tag{29}$$

for C_1 a constant depending on A, C, p and η only. We then use Young's inequality to get, from (29),

$$E(t) \le C_2\left((F(t))^{2(m-1)} + (F(t))^m\right). \tag{30}$$

At this end, we distinguish two cases:

1) $m = 2$. In this case, we have from (30)

$$E(t) \;\le\; 2C_2F^2(t) \;\le\; C_3F^2(t) \;\le\; \frac{C_3}{c_0}\left(E(t) - E(t+1)\right). \tag{31}$$

Lemma 2.1 then yields

$$E(t) \le E(0)e^{-k[t-1]^+}, \quad k = \ln\left(\frac{C_3}{C_3 - c_0}\right). \tag{32}$$

2) $m > 2$. In this case, we note that, by (19), we have

$$F^m(t) \leq \frac{E(t)}{c_0} \leq \frac{E(0)}{c_0}.$$

Therefore (30) gives

$$\begin{aligned} E(t) &\leq C_2\left((F(t))^{2(m-2)} + (F(t))^{m-2}\right)F^2(t) \\ &\leq C_3\left((\frac{E(0)}{c_0})^{\frac{2(m-2)}{m}} + (\frac{E(0)}{c_0})^{\frac{m-2}{m}}\right)F^2(t) \\ &\leq C_4F^2(t); \end{aligned} \tag{33}$$

hence

$$E^{\frac{m}{2}}(t) \leq C_5F^m(t) \leq \frac{C_5}{c_0}\left(E(t) - E(t+1)\right). \tag{34}$$

Again Lemma 2.1 for

$$r = \frac{m-2}{2} > 0, \tag{35}$$

gives

$$E(t) \leq \left\{E(0)^{-(\frac{m-2}{2})} + \frac{C_5}{c_0}\frac{m-2}{2}[t-1]^+\right\}^{-\frac{2}{m-2}}$$

This completes the proof.

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Said Berrimi
Math. Department
University of Setif
Setif, Algeria
e-mail: `berrimi@yahoo.fr`

Salim A. Messaoudi
Mathematical Sciences Department
KFUPM, Dhahran 31261
Saudi Arabia
e-mail: `messaoud@kfupm.edu.sa`