Uniform decay in a Timoshenko-type system with past history

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1 Introduction

Timoshenko model for a thick beam (1921):

$$\begin{aligned} \rho u_{tt} &= (K(u_x - \varphi))_x \\ I_\rho \varphi_{tt} &= (EI\varphi_x)_x + K(u_x - \varphi) \end{aligned} \quad (.1)$$

t : time variable
x : space variable along the beam of length L
u : transverse displacement
φ : rotation angle of the filament of the beam
ρ : density (the mass per unit length)
I_ρ : the polar moment of inertia of a cross section
E :Young′s modulus of elasticity
I : the moment of inertia of a cross section
K : the shear modulus.

Recent Literature:
* Raposo *et al.* used two linear frictional dampings and established an exponential decay result

* Kim and Renardy considered (1) with two boundary controls of the form

$$\begin{aligned} K\varphi(L,t) - K\frac{\partial u}{\partial x}(L,t) &= \alpha\frac{\partial u}{\partial t}(L,t) \\ EI\frac{\partial \varphi}{\partial x}(L,t) &= -\beta\frac{\partial \varphi}{\partial t}(L,t) \end{aligned}$$

- established an exponential decay result
- provided numerical estimates to the eigenvalues of the operator associated with system (1).

* Yan generalized the above result to nonlinear boundary conditions

$$\begin{aligned} K(\varphi(L,t) - \frac{\partial u}{\partial x}(L,t)) &= f_1(\frac{\partial u}{\partial t}(L,t)) \\ -EI\frac{\partial \varphi}{\partial x}(L,t) &= f_2(\frac{\partial \varphi}{\partial t}(L,t)), \end{aligned}$$

f_1, f_2 are functions with polynomial growth near the origin

* Ammar-Khodja *et al.* studied

$$
\begin{aligned}
&\alpha u_{tt} = (\beta(u_x + \varphi))_x \\
&\gamma \varphi_{tt} = (\delta \varphi_x)_x - K(u_t + \varphi), \\
&u(0,t) = u(L,t) = 0,\ \varphi_x(0,t) = c\varphi_t(0,t), \\
&\varphi_x(L,t) = -d\varphi_t(L,t)
\end{aligned}
\tag{.2}
$$

$\alpha(x), \beta(x), \gamma(x), \delta(x)$ are positive C^1-functions.
They proved

- exponential stability of (2) holds if and only if

$$\frac{\delta}{\gamma} = \frac{\beta}{\alpha} \qquad \text{on } (0,L)$$

- otherwise only the asymptotic stability holds

* Soufyane and Wehbe considered

$$
\begin{aligned}
&\rho u_{tt} = (K(u_x - \varphi))_x \\
&I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi) - b(x)\varphi_t \\
&u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0
\end{aligned}
\tag{.3}
$$

$b(x)$: positive,continuous, satisfying

$$b(x) \;\geq\; b_0 > 0, \quad \forall\, x \;\in\; [a_0, a_1] \subset [0, L]$$

They proved

- exponential stability of (3) holds if and only if $\frac{\delta}{\gamma} = \frac{\beta}{\alpha}$
- otherwise only the asymptotic stability holds

* Rivera and Racke obtained same result, allowing b to change sign

* Ammar-Khodja *et al.* considered a Timoshenko-type system with memory

$$\begin{aligned} &\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ &\rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0 \end{aligned}$$

in $(0, L)\times(0, +\infty)$, together with homogeneous boundary conditions. They proved, for $\frac{\delta}{\gamma} = \frac{\beta}{\alpha}$

- an exponential decay if g decays in an exponential rate
- polynomially if g decays in a polynomial rate
- They also required some extra technical conditions on both g' and g''

* Guesmia and Messaoudi obtained the same uniform decay results without imposing those extra technical

conditions on g' and g''

* Recently, Messaoudi and Mustafa:
- allowed more general relaxation functions
- obtained a more general decay result
- **exponential** and the **polynomial** decay results are only **special cases**

* Rivera and Racke treated a nonlinear Timoshenko-type system of the form

$$\begin{aligned} &\rho_1\varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0 \\ &\rho_2\psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x, \psi) + d\psi_t = 0 \end{aligned} \tag{.4}$$

- gave an alternative proof for a necessary and sufficient condition for exponential stability in the linear case
- proved a polynomial stability in general
- investigated the global existence of small smooth solutions and exponential stability in the nonlinear case

* Recently, Fernández Sare and Rivera considered a Timoshenko-type system with a past history of the form

$$\begin{array}{l}\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^\infty g(s)\psi_{xx}(t-s,.)ds \\ +K(\varphi_x+\psi) = 0\end{array} \tag{.5}$$

ρ_1, ρ_2, K, b: positive constants
g: positive twice differentiable function satisfying, for some constants $k_0, k_1, k_2 > 0$,

$$\begin{array}{c} g(t) \; > \; 0, \quad -k_0 g(t) \leq g'(t) \leq -k_1 g(t), \\ \widehat{b} \; = \; b - \displaystyle\int_0^\infty g(s)\,ds > 0, \quad |g''| \leq k_2 g(t) \end{array} \tag{.6}$$

and showed

- the system is exponentially stable if and only if the wave speeds are equal
- the solution decays polynomially for the case of different wave speeds.

More

- Rivera and Racke (Timoshenko systems in thermoelasticity)
- Fernández Sare and Racke (nonexponential decay for Timoshenko systems in thermoelasticity with sec-

ond sound)

- Messaoudi *et al.* (Timoshenko systems in thermoelasticity with second sound)

- Messaoudi and Said-Houari (Timoshenko systems in thermoelasticity type III)

- Alabau-Boussouira (Timoshenko systems with nonlinear boundary conditions)

- Messaoudi and Mustafa (several results)

Present work System (5) with

$$\begin{array}{l} \varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0 \\ \varphi(.,0)=\varphi_0,\ \varphi_t(.,0)=\varphi_1, \\ \psi(.,0)=\psi_0,\ \psi_1(.,0)=\Psi_1 \end{array} \tag{.7}$$

where g is a differentiable function satisfying, for a positive constant k_0 and $1 \leq p < 3/2$, the following conditions

$$\begin{array}{l} g(t) \ > \ 0, \qquad \widehat{b}=b-\int_0^\infty g(s)\,ds>0 \\ g'(t) \ \leq \ -k_0 g^p(t) \end{array} \tag{.8}$$

Remark 1. Under condition (8): $1 \leq p < 3/2$,

$$G_0 = \int_0^\infty g^{1/2}(s)\,ds < \infty, \qquad (.9)$$
$$G_p = \int_0^\infty g^{2-p}(s)\,ds < \infty$$

Objective: show that, for

$$\frac{\rho_1}{K} = \frac{\rho_2}{b}$$

- the energy decays exponentially if $p = 1$
- the energy decays polynomially if $p > 1$

$$\frac{\rho_1}{K} \neq \frac{\rho_2}{b}$$

- the decay is in the rate of $t^{-1/(2p-1)}$ if initial data are regular enough.

Set

$$\eta^t(x,s) = \psi(x,t) - \psi(x,t-s),\ s \geq 0; \qquad (.10)$$

consequently

$$\eta^t(x,0) = 0,\ \forall t \geq 0$$
$$\eta^t(0,s) = \eta^t(1,s) = 0,\ \forall s,t \geq 0 \qquad (.11)$$
$$\eta^0(x,s) = \eta_0(x,s),\ \forall s \geq 0.$$

and

$$\eta_t^t(x,s) + \eta_s^t(x,s) = \psi_t(x,t). \qquad (.12)$$

2 Uniform decay $\frac{\rho_1}{K} = \frac{\rho_2}{b}$

Combining (5), (7), (11), (12), we obtain the following system

$$\begin{array}{l} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_{2.}\psi_{tt} - \widehat{b}\psi_{xx} + \int_0^\infty g(s)\,\eta^t_{xx}(x, t-s)\,ds \\ +K(\varphi_x + \psi) = 0 \\ \eta^t_t(x,s) + \eta^t_s(x,s) - \psi_t(x,t) = 0 \end{array} \quad (.13)$$

in $(0,1) \times (0,\infty)$, together with

$$\begin{aligned} \varphi(0,t) &= \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 \\ \eta^t(0,s) &= \eta^t(1,s) = 0,\ t \geq 0 \qquad (.14) \\ \varphi(.,0) &= \varphi_0,\ \varphi_t(.,0) = \varphi_1,\ \psi(.,0) = \psi_0, \\ \psi_1(.,0) &= \psi_1, \eta^t(x,0) = 0 \end{aligned}$$

The first-order energy is

$$\begin{aligned} E(t) &= E_1(\varphi, \psi, \eta^t) \qquad (.15) \\ &= \frac{1}{2}\int_0^1 \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + K\left|\varphi_x + \psi\right|^2 + \widehat{b}\psi_x^2\right) dx \\ &\quad + \frac{1}{2}\int_0^1 \int_0^\infty g(s)\left|\eta^t_x(s)\right|^2 ds dx. \end{aligned}$$

Theorem 2.1 *Suppose that* (8) *and*

$$\frac{\rho_1}{K} = \frac{\rho_2}{b} \tag{.16}$$

hold and let

$$\varphi_0, \psi_0 \in H_0^1(0,1), \quad \eta_0^t \in L_g^2\left(\mathbb{R}^+, H_0^1(0,1)\right),$$
$$\varphi_1, \psi_1 \in L^2(0,1).$$

Then, there exist two positive constants C *and* ξ*, such that*

$$E(t) \le Ce^{-\xi t}, \qquad p = 1 \tag{.17}$$

$$E(t) \le \frac{C}{(t+1)^{1/(p-1)}} \qquad p > 1, \tag{.18}$$

.

Lemma 2.1.

$$\frac{dE(t)}{dt} = \frac{1}{2}\int_0^1 \int_0^\infty g'(s)\left|\eta_x^t(s)\right|^2 dsdx \le 0. \tag{.19}$$

Lemma 2.2.

$$\left(\int_0^1 \int_0^\infty g(s)\left|\eta_x^t(s)\right|^2 dsdx\right)^{2p-1} \tag{.20}$$
$$\le C_0 \int_0^1 \int_0^\infty g^p(s)\left|\eta_x^t(s)\right|^2 dsdx.$$

for a constant $C_0 > 0$.

Lemma 2.3. *For* $1 \leq p \leq 3/2$*,*

$$\int_0^1 \left(\int_0^\infty g(s)\, \eta_x^t(s)\, ds \right)^2 dx \leq G_p \int_0^1 \int_0^\infty g^p(s) \left| \eta_x^t(s) \right|^2 ds dx \qquad (.21)$$

let

$$I_1 := \int_0^1 (\rho_2 \psi_t \psi + \rho_1 \varphi_t \omega)\, dx. \qquad (.22)$$

where ω satisfies

$$-\omega_{xx} = \psi_x, \quad \omega(0) = \omega(1) = 0$$

Lemma 2.4 *For any* $\varepsilon_1, \lambda_1 > 0$

$$\begin{aligned} \frac{dI_1(t)}{dt} \leq & \left(-\widehat{b} + \lambda_1 \right) \int_0^1 \psi_x^2 dx \qquad (.23) \\ & + \varepsilon_1 \rho_1 \int_{0t}^1 \varphi_t^2 dx + \left(\rho_2 + \frac{\rho_1}{4\varepsilon_1} \right) \int_0^1 \psi_t^2 dx \\ & + \frac{G_p}{4\lambda_1} \int_0^1 \int_0^\infty g^p(s) \left| \eta_x^t(s) \right|^2 ds dx. \end{aligned}$$

set

$$I_2 := -\rho_2 \int_0^1 \psi_t(x,t) \int_0^\infty g(s)\,\eta^t(s)\,dsdx \quad (.24)$$

Lemma 2.5 *For any* $\varepsilon_2 > 0$,

$$\begin{aligned}\frac{dI_2(t)}{dt} \le & -\frac{\rho_2 g_0}{2}\int_0^1 \psi_t^2 dx \quad (.25)\\ & +\varepsilon_2 \widehat{b}^2 \int_0^1 \psi_x^2 dx + \varepsilon_2 K^2 \int_0^1 (\varphi_x+\psi)^2 dx\\ & +G_p\left(1+\frac{1}{2\varepsilon_2}\right)\int_0^1\int_0^\infty g^p(s)\left|\eta_x^t(s)\right|^2 dsdx\\ & -\frac{g(0)}{2\rho_2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(s)\right|^2 dsdx,\end{aligned}$$

where $g_0 = \int_0^\infty g(s)\,ds$

Introduce

$$\begin{aligned}J(t) := {}& \rho_2\int_0^1 \psi_t(\varphi_x+\psi)\,dx + \frac{\rho_1\widehat{b}}{K}\int_0^1 \psi_x\varphi_t dx\\ & +\frac{\rho_1}{K}\int_0^1 \varphi_t(t)\int_0^\infty g(s)\,\widehat{\eta}_x^t(s)\,dsdx \quad (.26)\end{aligned}$$

Lemma 2.6. *For* $\varepsilon_3 > 0$, *we conclude*

$$\frac{dJ(t)}{dt} \le \left[\varphi_x\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\right)\right]_{x=0}^{x=1} (.27)$$

$$-K\int_0^1 (\varphi_x+\psi)^2\,dx$$
$$+\rho_2\int_0^1 \psi_t^2 dx+\varepsilon_3\int_0^1 \varphi_t^2 dx$$
$$-g(0)C(\varepsilon_3)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(s)\right|^2 dsdx.$$

Proof. Differentiating $J(t)$, using equations (13), we find

$$\begin{aligned}\frac{dJ(t)}{dt} &= \int_0^1 (\varphi_x+\psi)\left[\widehat{b}\psi_{xx}+\int_0^\infty g(s)\,\eta_{xx}^t(x,s)\,ds\right] dx \\ &\quad -K\int_0^1 (\varphi_x+\psi)^2\,dx \\ &\quad +\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^1 \psi_{tx}\varphi_t dx \\ &\quad +\widehat{b}\int_0^1 \psi_x(\varphi_x+\psi)_x\,dx+\rho_2\int_0^1 \psi_t^2 dx \\ &\quad -\frac{\rho_1}{K}\int_0^1 \varphi_t(t)\int_0^\infty g(s)\,\eta_{sx}^t(s)\,dsdx \\ &\quad +\int_0^1 (\varphi_x+\psi)_x\int_0^\infty g(s)\,\eta_x^t(s)\,dsdx\end{aligned} \tag{.28}$$

Exploiting (16), we conclude

$$\begin{aligned}\frac{dJ(t)}{dt} &= -K\int_0^1 (\varphi_x+\psi)^2\,dx + \rho_2\int_0^1 \psi_t^2 dx \\ &\quad +\frac{\rho_1}{K}\int_0^1 \varphi_t\int_0^\infty g'(s)\,\eta_x^t(s)\,dsdx \\ &\quad +\left[\widehat{b}\varphi_x\psi_x\right]_{x=0}^{x=1} \\ &\quad +\left[\varphi_x(x,t)\int_0^\infty g(s)\,\eta_x^t(x,s)\right]_{x=0}^{x=1}\end{aligned}$$

Young's inequality gives (27).

Define

$$\mathcal{K}(t) := -\rho_1\int_0^1 \varphi_t\varphi dx - \rho_2\int_0^1 \psi_t\psi dx.$$

Lemma 2.8. *For any* $\varepsilon_3 > 0$, *we get*

$$\begin{aligned}\frac{d}{dt}\mathcal{K}(t) &\le -\rho_1\int_0^1 \varphi_t^2 dx - \rho_2\int_0^1 \psi_t^2 dx \qquad (.29)\\ &\quad +\left(\widehat{b}+\varepsilon_3\right)\int_0^1 \psi_x^2 dx + K\int_0^1 (\varphi_x+\psi)^2\,dx \\ &\quad +\frac{G_p}{4\varepsilon_3}\int_0^1\int_0^\infty g^p(s)\left|\eta_x^t(s)\right|^2 dsdx\end{aligned}$$

Proof of Theorem 2.1

Let the Lyapunov functional $\mathcal{L}$:

$$\mathcal{L}(t) := NE(t) + N_1 I_1 + N_2 I_2 + J(t) + \frac{\varepsilon_3}{K}\int_0^1 \rho_1 q\varphi_t\varphi_x dx + \mu\mathcal{K}(t) + \frac{1}{4\varepsilon_3}\int_0^1 \rho_2 q(x)\psi_t\left(\widehat{b}\psi_x + \int_0^\infty g(s)\eta_x^t(s)\,ds\right)dx \quad (.30)$$

for

$$q(x) = 2 - 4x, \qquad x \in [0,1]$$

and $N, N_1, N_2, \mu, \varepsilon_3$ positive constants

Use of lemmas and careful chose of $N, N_1, N_2, \mu, \varepsilon_3$ lead to

$$\frac{d}{dt}\mathcal{L}(t) \le -\sigma_1\left[\int_0^1\left(\psi_x^2 + \psi_t^2 + \varphi_t^2 + (\varphi_x + \psi)^2\right)dx + \int_0^1\int_0^\infty g^p(s)\left|\eta_x^t(s)\right|^2 dsdx\right] \quad (.31)$$

and, for N large enough,

$$\mathcal{L}(t) \backsim E(t) \quad (.32)$$

Two cases:

Case 1: $p = 1$. (31) and (32) $\Longrightarrow$

$$\frac{d}{dt}\mathcal{L}(t) \le -\xi\mathcal{L}(t) \qquad (.33)$$

Simple integration $\Longrightarrow$

$$E(t) \le Ce^{-\xi t}, \qquad p = 1 \qquad (.34)$$

Case 2. $p > 1$.

$$\begin{aligned} E^{2p-1}(t) &\le C(E(0))^{2p-2}\int_0^1\left(\varphi_t^2+\psi_t^2+|\varphi_x+\psi|^2+\psi_x^2\right)dx \\ &\quad + C\left[\int_0^1\int_0^\infty g(s)\left|\eta_x^t(s)\right|^2 dsdx\right]^{2p-1} \qquad (.35) \\ &\le C(E(0))^{2p-2}\int_0^1\left(\varphi_t^2+\psi_t^2+|\varphi_x+\psi|^2+\psi_x^2\right)dx \\ &\quad + CC_0\int_0^1\int_0^\infty g^p(s)\left|\eta_x^t(s)\right|^2 dsdx. \end{aligned}$$

(31), (32) and (35)$\Longrightarrow$

$$\mathcal{L}'(t) \le -cE^{2p-1}(t) \le -c\mathcal{L}^{2p-1}(t) \qquad (.36)$$

Simple integration $\Longrightarrow$

$$\mathcal{L}(t) \leq \frac{C}{(t+1)^{1/(2p-2)}} \tag{.37}$$

To obtain

$$E(t) \ \leq \ \frac{C}{(t+1)^{1/(p-1)}} \qquad p > 1, \tag{.38}$$

observe

$$\int_0^1 \int_0^t g(s) \left|\eta_x^t(s)\right|^2 dsdx \tag{.39}$$

$$= \int_0^1 \int_0^t g(s) \left|\eta_x^t(s)\right|^{\frac{2}{p}} \left|\eta_x^t(s)\right|^{\frac{2(p-1)}{p}} dsdx$$

$$\leq \left(\int_0^1 \int_0^t \left|\eta_x^t(s)\right|^2 dsdx\right)^{(p-1)/p}$$

$$\times \left(\int_0^1 \int_0^t g^p(s) \left|\eta_x^t(s)\right|^2 dsdx\right)^{1/p}$$

and use $\eta_x^t(x,s) = \psi_x(x,t) - \psi_x(x,t-s)$ to conclude that

$$\int_0^t \int_0^1 \left|\eta_x^t(s)\right|^2 dxds \leq 2 \int_0^t \int_0^1 \left|\psi_x(x,t)\right|^2 dxds$$

$$+2 \int_0^t \int_0^1 \left|\psi_x(x,t-s)\right|^2 dxds$$

$$
\begin{aligned}
&\leq\ 2t\int_0^1 \left|\psi_x(x,t)\right|^2 dx+\frac{4}{\widehat{b}}\int_0^t E(t-s)ds\\
&\leq\ \frac{Ct}{(t+1)^{1/(2p-2)}}+C\int_0^t \frac{ds}{(t-s+1)^{1/(2p-2)}}\\
&\leq\ \frac{C}{(t+1)^{(3-2p)/(2p-2)}}+\frac{2p-2}{3-2p}C\left[1-\frac{1}{(t+1)^{(3-2p)/(2p-2)}}\right]\\
&\leq\ \Pi, \qquad p<3/2,
\end{aligned}
$$

where Π is a constant independent of t. Hence, we get

$$
\int_0^\infty\int_0^1 \left|\eta_x^t(s)\right|^2 dxds\leq\Pi
$$

Consequently:

$$
\begin{aligned}
&\int_0^1\int_0^\infty g(s)\left|\eta_x^t(s)\right|^2 dsdx\\
&\quad\leq\ \Pi^{(p-1)/p}\left(\int_0^1\int_0^\infty g^p(s)\left|\eta_x^t(s)\right|^2 dsdx\right)^{1/p}
\end{aligned}
$$

or

$$
\begin{aligned}
&\left(\int_0^\infty\int_0^1 g(s)\left|\eta_x^t(s)\right|^2 dxds\right)^p\\
&\quad\leq\ C\int_0^\infty\int_0^1 g^p(s)\left|\eta_x^t(s)\right|^2 dxds \qquad (.40)
\end{aligned}
$$

Similarly to (35), we obtain

$$
\begin{aligned}
E^p(t) &\le C\int_0^1\left(\varphi_t^2+\psi_t^2+|\varphi_x+\psi|^2+\psi_x^2\right)dx \\
&\quad +C\left[\int_0^1\int_0^\infty g(s)\left|\eta_x^t(s)\right|^2 dsdx\right]^p \quad (.41)\\
&\le C\int_0^1\left(\varphi_t^2+\psi_t^2+|\varphi_x+\psi|^2+\psi_x^2\right)dx \\
&\quad +C\int_0^1\int_0^\infty g^p(s)\left|\eta_x^t(s)\right|^2 dsdx.
\end{aligned}
$$

A combination of (31), (32), and (41) yields

$$
\mathcal{L}'(t) \le -cE^p(t) \le -c\mathcal{L}^p(t) \qquad (.42)
$$

Simple integration $\Longrightarrow$

$$
\mathcal{L}(t) \le \frac{C}{(t+1)^{1/(p-1)}} \qquad (.43)
$$

3 Polynomial decay $\frac{\rho_1}{K} \neq \frac{\rho_2}{b}$

This is, in fact, more realistic from the view point of physics

If the initial data are regular enough the solution energy $E(t)$

- decays in the rate of $1/t$ when the relaxation function g decays exponentially
- decays in the rate of $t^{-1/(2p-1)}$ when the relaxation function decays polynomially.

Let's define the second-order energy by

$$\begin{aligned} E_2(t) &= E_1(\varphi_t, \psi_t, \eta_t^t) && (.44) \\ &= \frac{1}{2}\int_0^1 \left(\rho_1\varphi_{tt}^2 + \rho_2\psi_{tt}^2 + K\left|\varphi_{xt} + \psi_t\right|^2 + \widehat{b}\psi_{xt}^2\right) dx \\ &\quad + \frac{1}{2}\int_0^1\int_0^\infty g(s)\left|\eta_{xt}^t(s)\right|^2 dsdx. \end{aligned}$$

Theorem 3.1 *Suppose that*

$$\frac{\rho_1}{K} \neq \frac{\rho_2}{b} \qquad (.45)$$

and let

$$\begin{aligned} \varphi_0, \psi_0 &\in H^2(0,1) \cap H_0^1(0,1), && (.46) \\ \eta_0^t &\in L_g^2\left(\mathbb{R}^+, H^2(0,1) \cap H_0^1(0,1)\right), \end{aligned}$$

$$\varphi_1, \psi_1 \in H_0^1(0,1).$$

Then there exists a positive constants C*, such that,* $\forall t \geq 0$,

$$E(t) \leq Ct^{-1/(2p-1)} \qquad p \geq 1. \tag{.47}$$

Need two lemmas.

Lemma 3.1. *We have*

$$\begin{aligned} \frac{dE_2(t)}{dt} &= -\frac{1}{2}\int_0^1 \int_0^\infty g'(s)\left|\eta_{tx}^t(s)\right|^2 dsdx \\ &\leq 0. \end{aligned} \tag{.48}$$

Lemma 3.2. *For* $\varepsilon_3 > 0$*, have*

$$\begin{aligned} \frac{dJ(t)}{dt} \leq & \left[\varphi_x\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\right)\right]_{x=0}^{x=1} \\ & -K\int_0^1 (\varphi_x + \psi)^2\,dx \\ & +\rho_2\int_0^1 \psi_t^2 dx + 2\varepsilon_3\int_0^1 \varphi_t^2 dx \\ & -g(0)C(\varepsilon_3)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(s)\right|^2 dsdx \end{aligned} \tag{.49}$$

$$+C(\varepsilon_3)\int_0^1\int_0^\infty g^p(s)\left|\eta_{tx}^t(s)\right|^2 dsdx$$

Proof. Similarly to proof of Lemma 2.7

$$\begin{aligned}\frac{dJ(t)}{dt} &= \int_0^1(\varphi_x+\psi)\left[\widehat{b}\psi_{xx}+\int_0^\infty g(s)\,\eta_{xx}^t(x,s)\,ds\right]dx \\ &-K\int_0^1(\varphi_x+\psi)^2\,dx+\rho_2\int_0^1\psi_t^2dx \\ &+\widehat{b}\int_0^1\psi_x(\varphi_x+\psi)_x\,dx+\left(\frac{\rho_1 b}{K}-\rho_2\right)\int_0^1\psi_{tx}\varphi_t dx \\ &-\frac{\rho_1}{K}\int_0^1\varphi_t(t)\int_0^\infty g(s)\,\eta_{xs}^t(s)\,dsdx \\ &+\int_0^1(\varphi_x+\psi)_x\int_0^\infty g(s)\,\eta_x^t(s)\,dsdx\end{aligned} \tag{.50}$$

Handle $\int_0^1\psi_{tx}\,\varphi_t dx$ as follows

$$\begin{aligned}\int_0^1\psi_{tx}\varphi_t dx &= \frac{1}{g_0}\int_0^1\left(\int_0^\infty g(s)\,\eta_{tx}^t(s)\,ds\right)\varphi_t dx \\ &-\frac{1}{g_0}\int_0^1\left(\int_0^\infty g'(s)\,\eta_x^t(s)\,ds\right)\varphi_t dx,\end{aligned} \tag{.51}$$

where $g_0=\int_0^\infty g(s)\,ds.$

Proof of Theorem 3.1.

Lyapunov functional $\mathcal{L}$:

$$\begin{aligned}\mathcal{L}(t) \,:\, &= N\left[E_1(t) + E_2(t)\right] + N_1 I_1 + \mu\mathcal{K}(t) \\ &+ N_2 I_2 + J(t) + \frac{\varepsilon_3}{K}\int_0^1 \rho_1 q\varphi_t\varphi_x dx \\ &+ \frac{1}{4\varepsilon_3}\int_0^1 \rho_2 q(x)\psi_t\left(\widehat{b}\psi_x + \int_0^\infty g(s)\eta_x^t(s)ds\right)dx\end{aligned} \quad (.52$$

Use of lemmas and careful chose of $N, N_1, N_2, \mu, \varepsilon_3$ lead to

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \,\le\, -\sigma_2 &\left[\int_0^1 \left(\psi_x^2 + \psi_t^2 + \varphi_t^2 + (\varphi_x + \psi)^2\right)dx\right. \\ &+ \int_0^1\int_0^\infty g^p(s)\left|\eta_x^t(s)\right|^2 dsdx \\ &\left.+ \int_0^1\int_0^\infty g^p(s)\left|\eta_{tx}^t(s)\right|^2 dsdx\right]\end{aligned} \quad (.53)$$

Case 1. $p = 1$.

$$\frac{d}{dt}\mathcal{L}(t) \le -\alpha E(t) \quad (.54)$$

Direct integration gives

$$\alpha\int_0^t E(s)ds \,\le\, \mathcal{L}(0) - \mathcal{L}(t) \le \mathcal{L}(0) \quad (.55)$$

Using (52), one can find $\sigma_3 > 0$,

$$\mathcal{L}(0) \leq \sigma_3 (E_1(0) + E_2(0)) . \tag{.56}$$

(55) and (56) $\Longrightarrow$

$$\int_0^t E(s)ds \leq C(E_1(0) + E_2(0)) \tag{.57}$$

But

$$\frac{d}{dt}(tE(t)) = E(t) + t\frac{d}{dt}E(t) \leq E(t)$$

Simple integration gives

$$tE(t) \leq \int_0^t E(s)ds \leq C(E_1(0) + E_2(0))$$

So

$$E(t) \leq \frac{C}{t}(E_1(0) + E_2(0))$$

Case 2. $p > 1$. By using (36), we obtain

$$\frac{d\mathcal{L}}{dt} \le -cE^{2p-1}(t)$$

which implies

$$\int_0^t E^{2p-1}(s)ds \le c\mathcal{L}(0), \qquad \forall t \ge 0.$$

On the other hand

$$\begin{aligned} \frac{d}{dt}\left(tE^{2p-1}(t)\right) &= E^{2p-1}(t) + (2p-1)tE^{2p-2}\frac{d}{dt}E(t) \\ &\le E^{2p-1}(t). \end{aligned}$$

Simple integration $\Longrightarrow$

$$tE^{2p-1}(t) \le \int_0^t E^{2p-1}(s)ds \le c\mathcal{L}(0)$$

Thus

$$E(t) \le Ct^{-1/(2p-1)}$$

Remark 3.1. Note that this result generalizes the result of Rivera and Fernandez to the case $p > 1$. Moreover, our result is established without any condition on g''.