Section 9.3

problem #1

$$\begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 - 4 = \lambda^2 - 5\lambda = 0 \Rightarrow \lambda = 0, 5.$$

For $\boxed{\lambda = 0}$

$$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right) \xrightarrow[2R2+R1]{R1/4} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 = \frac{1}{2} x_2$$

An eigenvector is $E_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For $\boxed{\lambda = 5}$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \xrightarrow[R2-2R1]{-R1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow x_1 = -2x_2$$

So an eigenvector is $E_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

$$E_1 \cdot E_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -2 + 2 = 0 \Rightarrow E_1 \perp E_2.$$

Let's find the unit vectors parallel to E_1 and E_2 respectively

$$U_1 = \frac{E_1}{\|E_1\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad U_2 = \frac{E_2}{\|E_2\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

An orthogonal matrix P which diagonalize A is

$$P = \begin{pmatrix} U_1 & U_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

Note that $\det P = 1$.

problem # 6

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2-\lambda & 0 \end{vmatrix} + (2-\lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= -(2-\lambda) + (2-\lambda)\left[\lambda^2 - 2\lambda - 1\right]$$

$$= (2-\lambda)\left[-1 + \lambda^2 - 2\lambda - 1\right]$$

$$= (2-\lambda)(\lambda^2 - 2\lambda - 2) = 0 \Rightarrow \lambda = 2, 1 \pm \sqrt{3}.$$

The eigenvalues are real.

Eigenvectors

$\boxed{\lambda = 2}$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[\substack{2R_2 + R_1 \\ 2R_3 + R_1}]{-R_1} \begin{pmatrix} +2 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow[\substack{R_2 \\ R_3 - R_2}]{R_1 + R_2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So $x_1 = 0$ and $x_2 = -x_3$

An eigenvector will be $E_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

$\boxed{\lambda = 1 - \sqrt{3}}$

$$\begin{pmatrix} -1+\sqrt{3} & 1 & 1 \\ 1 & 1+\sqrt{3} & 0 \\ 1 & 0 & 1+\sqrt{3} \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1+\sqrt{3} & 0 \\ -1+\sqrt{3} & 1 & 1 \\ 1 & 0 & 1+\sqrt{3} \end{pmatrix} \xrightarrow[R_3 - R_1]{\substack{R_1 \\ R_2 + (1-\sqrt{3})R_1}}$$

$$\begin{pmatrix} 1 & 1+\sqrt{3} & 0 \\ 0 & -1 & 1 \\ 0 & -1-\sqrt{3} & 1+\sqrt{3} \end{pmatrix} \xrightarrow[\substack{-R_2 \\ R_3 - (1+\sqrt{3})R_2}]{R_1 + (1+\sqrt{3})R_2} \begin{pmatrix} 1 & 0 & 1+\sqrt{3} \\ 0 & +1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

So $x_1 = -(1+\sqrt{3})x_3$ and $x_2 = x_3$

An eigenvector will be $E_2 = \begin{pmatrix} -1-\sqrt{3} \\ 1 \\ 1 \end{pmatrix}$

$\boxed{\lambda = 1+\sqrt{3}}$

$$\begin{pmatrix} -1-\sqrt{3} & 1 & 1 \\ 1 & 1-\sqrt{3} & 0 \\ 1 & 0 & 1-\sqrt{3} \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1-\sqrt{3} & 0 \\ -1-\sqrt{3} & 1 & 1 \\ 1 & 0 & 1-\sqrt{3} \end{pmatrix} \xrightarrow[R_3 - R_1]{\substack{R_1 \\ R_2+(1+\sqrt{3})R_1}}$$

$$\begin{pmatrix} 1 & 1-\sqrt{3} & 0 \\ 0 & -1 & 1 \\ 0 & -1+\sqrt{3} & 1-\sqrt{3} \end{pmatrix} \xrightarrow[\substack{-R_2 \\ R_3 \bullet (1-\sqrt{3})R_2}]{R_1+(1-\sqrt{3})R_2} \begin{pmatrix} 1 & 0 & 1-\sqrt{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Similary, we have an eigenvector $E_3 = \begin{pmatrix} -1+\sqrt{3} \\ 1 \\ 1 \end{pmatrix}$

let's verify that all these eigenvector are orthogonal.

$E_1 \cdot E_2 = 0 - 1 + 1 = 0$; $E_1 \cdot E_3 = -1+1 = 0$

$E_2 \cdot E_3 = (-1-\sqrt{3})(-1+\sqrt{3}) + 1 + 1 = -2+1+1 = 0$

The matrix $M = (E_1, E_2, E_3)$ will not be orthogonal unless $\|E_1\| = 1$, $\|E_2\| = 1$, $\|E_3\| = 1$. This is not the case. So, an orthogonal transition matrix is

$$P = \begin{pmatrix} \frac{E_1}{\|E_1\|} & \frac{E_2}{\|E_2\|} & \frac{E_3}{\|E_3\|} \end{pmatrix}$$

$$= \begin{bmatrix} 0 & \frac{-1-\sqrt{3}}{\sqrt{6+2\sqrt{3}}} & (-1+\sqrt{3})/\sqrt{6-2\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 1/\sqrt{6+2\sqrt{3}} & \frac{1}{\sqrt{6-2\sqrt{3}}} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{6+2\sqrt{3}} & \frac{1}{\sqrt{6-2\sqrt{3}}} \end{bmatrix}.$$