Section 9.2

problem #4

$$\begin{vmatrix} -5-\lambda & 3 \\ 0 & 9-\lambda \end{vmatrix} = (-5-\lambda)(9-\lambda) = 0 \iff \lambda = -5, 9$$

So -5 and 9 are the eigenvalue. It is clear that A is diagonalizable since the eigenvalues are distinct.

Eigenvectors

$\boxed{\lambda = -5}$

$$\left(\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 14 & 0 \end{array}\right) \xrightarrow{R_2 - \frac{14R_1}{3}} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad x_2 = 0 \quad \Rightarrow \quad X = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\therefore \quad E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector.

$\boxed{\lambda = 9}$

$$\left(\begin{array}{cc|c} -14 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -\frac{3}{14} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad x_1 = \frac{3}{14} x_2$$

$$\therefore \quad X = \begin{pmatrix} 3/14\, x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3/14 \\ 1 \end{pmatrix}$$

$E_2 = \begin{pmatrix} 3 \\ 14 \end{pmatrix}$ is an eigenvector.

A matrix that diagonalizes A is

$$P = \begin{pmatrix} 1 & 3 \\ 0 & 14 \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{14} \begin{pmatrix} 14 & -3 \\ 0 & 1 \end{pmatrix}$$

Check that
$$P^{-1} A P = D = \begin{pmatrix} -5 & 0 \\ 0 & 9 \end{pmatrix}.$$

problem #7

$$\begin{vmatrix} -2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 0 \\ \cancel{0} & \cancel{0} & \cancel{-2-\lambda} \end{vmatrix} = +(-2-\lambda)\begin{vmatrix} -2-\lambda & 0 \\ 1 & -\lambda \end{vmatrix} = (2+\lambda)^2(-\lambda) = 0$$

$$\Rightarrow \lambda = -2, -2, 0.$$

For $\boxed{\lambda = -2}$

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \Rightarrow \begin{cases} x_1 + 3x_2 = 0 \Rightarrow \\ \quad x_1 = -3x_2 \\ \quad \text{and} \\ \quad x_3 = 0 \end{cases}$$

The solution is $X = \begin{pmatrix} -3x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$ an eigenvector is

$E = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$. Since $\lambda = -2$ gave us only one eigenvector and it is of multiplicity 2. Then A is not diagonalizable.

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problem 12

If A is diagonalizable then $P^{-1}AP = D$ or

$A = PDP^{-1}$, where $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \\ & & \ddots & \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$

$$D^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & & \\ & & \ddots & \\ 0 & \cdots & 0 & \lambda_n^k \end{pmatrix}$$

$$A^k = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cdots(\cancel{P}DP^{-1}) = PD^kP^{-1}$$

$$= P \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & & \\ & & \ddots & \\ 0 & \cdots & & \lambda_n^k \end{pmatrix} P^{-1}$$