On the control of solutions of a viscoelastic equation

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1 Introduction

Cavalcanti *et al.* studied

$$\begin{cases} u_{tt} - \Delta u + \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t \\ +|u|^\gamma u = 0, \ \gamma > 0, \ \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \ x \in \partial\Omega \ , t \geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{1}$$

$\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ bounded with $\partial\Omega$ regular,
$g \geq 0$ with

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \ t \geq 0$$

and $||g||_{L^1((0,\infty))}$ is small enough,
$a : \Omega \to \mathbb{R}^+$ such that

$$a(x) \geq a_0 > 0 \quad on \quad \phi \neq \omega \subset \Omega,$$

with ω satisfying some geometry restrictions.
- An exponential decay result obtained.
- This extends the result of Zuazua,with $g = 0$.
- Berrimi and Messaoudi obtained the same result under weaker conditions on g and ω.

- Cavalcanti *et al* considered

$$u_{tt} - k_0\Delta u + \int_0^t div[a(x)g(t-\tau)\nabla u(\tau)]d\tau$$

$$+b(x)h(u_t) + f(u) = 0, \ \ \text{in } \Omega \times (0,\infty),$$

under similar conditions on g and

$$a(x) + b(x) \geq \delta > 0, \forall \ x \in \Omega.$$

They established an exponential stability for g decaying exponentially and h linear and polynomial stability for g decaying polynomially and h nonlinear.
- Cavalcanti *et al* have also studied

$$
\begin{aligned}
&|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau \\
&-\gamma \Delta u_t = 0, \ \rho > 0, \qquad x \in \Omega, \qquad t > 0,
\end{aligned}
$$

A global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$ were established.
- This last result has been extended to a situation, where a source term is present, by Messaoudi and Tatar.
- Also, Messaoudi considered

$$
\begin{aligned}
&u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau \\
&+au_t|u_t|^m = b|u|^\gamma u, \quad \text{in } \Omega \times (0,\infty)
\end{aligned}
$$

and showed, under suitable conditions on g, that solutions with negative energy blow up in finite time if $\gamma > m$ and continue to exist if $m \geq \gamma$.

In the absence of the viscoelastic term ($g = 0$), the problem has been extensively studied and many results concerning global existence and nonexistence have been proved. For instance, for the problem

$$
\begin{cases}
u_{tt} - \Delta u + au_t|u_t|^m = b|u|^\gamma u, & \text{in } \Omega \times (0,\infty) \\
u(x,t) = 0, & x \in \partial\Omega \ , t \geq 0 \\
u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega,
\end{cases} \tag{2}
$$

$a, b, m, \gamma \geq 0$

- For $a = 0$, the source term $bu|u|^\gamma$, ($\gamma > 0$) causes finite time blow up of solutions with negative initial energy
- For $b = 0$, the damping term $au_t|u_t|^m$ assures global existence for arbitrary initial data.
- For $ab \neq 0$, in the linear damping case ($m = 0$), a blow up result established by Levine for solutions with negative initial energy.
- Georgiev and Todorova studied the nonlinear damping case ($m > 0$). They showed that solutions with negative energy continue to exist globally 'in time' if $m \geq \gamma$ and blow up in finite time if $\gamma > m$ and the initial energy is sufficiently negative.
- Messaoudi proved the blow up result for solutions with negative initial energy only.
- Results of same nature were established by Levine and Serrin, and Levine and Park,

Vitillaro, Messaoudi and Said-Houari, and others.

Our problem

$$\begin{cases} u_{tt} - \Delta u + \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau = 0, & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \ x \in \partial\Omega \ , t \geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{3}$$

Conditions

(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded $\mathcal{C}^1$ function such that

$$g(0) > 0, \qquad 1 - \int\limits_0^\infty g(s)ds = l > 0.$$

(G2) There exists a positive constant ξ such that

$$g'(t) \leq -\xi g(t), \ t \geq 0,$$

Proposition *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that* g *satisfies* $(G1)$. *Then problem* (3) *has a unique global solution*

$$\begin{aligned} u &\in C\left([0,\ \infty); H_0^1(\Omega)\right), \\ u_t &\in C\left([0,\ \infty); L^2(\Omega)\right) \end{aligned} \tag{4}$$

Remark Condition (G1) is necessary to guarantee the hyperbolicity of the system (3).

The modified energy

$$\begin{aligned} \mathcal{E}(t) \ :&= \frac{1}{2}\left(1 - \int\limits_0^t g(s)ds\right)||\nabla u(t)||_2^2 \\ &+ \frac{1}{2}||u_t||_2^2 + \frac{1}{2}(g \circ \nabla u)(t), \end{aligned} \tag{5}$$

where

$$(g \circ v)(t) = \int\limits_0^t g(t-\tau)||v(t) - v(\tau)||_2^2 d\tau. \tag{6}$$

Lemma *The modified energy satisfies*

$$\begin{aligned} \mathcal{E}'(t) &= \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)||\nabla u(t)||^2 \\ &\leq \frac{1}{2}(g' \circ \nabla u)(t) \leq 0. \end{aligned} \tag{7}$$

2 Exponential decay

Theorem *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that* g *satisfies* $(G1)$ *and* $(G2)$*. Then there exist positive constants* k *and* K *such that the solution given by* (2.2) *satisfies*

$$\mathcal{E}(t) \leq Ke^{-kt}, \quad \forall t \geq t_0 > 0.$$

Proof We define

$$F(t) := \mathcal{E}(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t) \tag{8}$$

where ε_1 and ε_2 are positive constants to be specified later and

$$\begin{aligned} \Psi(t) &:= \int_\Omega u u_t dx \\ \chi(t) &:= -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx. \end{aligned}$$

It is straightforward to see that for ε_1 and ε_2 so small, we have

$$\alpha_1 F(t) \leq \mathcal{E}(t) \leq \alpha_2 F(t), \tag{9}$$

holds for two positive constants α_1 and α_2.

Direct, but lengthy, estimates give

$$\Psi'(t) \leq \int_\Omega u_t^2 dx - \frac{l}{2}\int_\Omega |\nabla u|^2 dx + \frac{(1-l)}{2l}(g \circ \nabla u)(t). \tag{10}$$

Similarly

$$\begin{aligned} \chi'(t) \leq\ & \delta\{1 + 2(1-l)^2\}||\nabla u||_2^2 \\ & + (2\delta + \frac{1}{2\delta})(1-l)(g \circ \nabla u)(t) \\ & + \frac{g(0)}{4\delta} C_p(-(g' \circ \nabla u)(t)) \\ & + (\delta - \int_0^t g(s)ds)\int_\Omega u_t^2 dx. \end{aligned} \tag{11}$$

Since $g(0) > 0$ then there exists $t_0 > 0$ such that

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0, \qquad \forall t \geq t_0. \tag{12}$$

A combination of (10) - (12) leads to

$$\begin{aligned} F'(t) &\leq -[\varepsilon_2\{g_0-\delta\}-\varepsilon_1]\int\limits_\Omega u_t^2dx \\ &\quad -\left[\frac{\varepsilon_1 l}{2}-\varepsilon_2\delta\{1+2(1-l)^2\}\right]||\nabla u||_2^2 \\ &\quad +\left[\frac{1}{2}-\frac{\varepsilon_1(1-l)}{2\xi l}-\varepsilon_2\{\frac{g(0)}{4\delta}C_p \right. \\ &\quad \left. +\frac{(1-l)}{\xi}(2\delta+\frac{1}{2\delta})\}\right](g'\circ\nabla u)(t). \end{aligned} \tag{13}$$

Choose δ so small that

$$\begin{aligned} g_0-\delta &> \frac{1}{2}g_0 \\ \frac{1}{l}\delta\{1+2(1-l)^2\} &< \frac{1}{8}g_0. \end{aligned}$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{1}{4}g_0\varepsilon_2<\varepsilon_1<\frac{1}{2}g_0\varepsilon_2 \tag{14}$$

will make

$$\begin{aligned} k_1 &= \varepsilon_2\{g_0-\delta\}-\varepsilon_1>0 \\ k_2 &= \frac{\varepsilon_1 l}{2}-\varepsilon_2\delta\{1+2(1-l)^2\}>0. \end{aligned}$$

Pick ε_1 and ε_2 so small that (9) and (14) remain valid and

$$\frac{1}{2}-\frac{\varepsilon_1(1-l)}{2\xi l}-\varepsilon_2\{\frac{g(0)}{4\delta}C_p+\frac{(1-l)}{\xi}(2\delta+\frac{1}{2\delta})\}>0$$

Therefore (13) becomes

$$F'(t)\leq-\beta\mathcal{E}(t), \qquad \forall t\geq t_0. \tag{15}$$

We then use (9) to arrive at

$$F'(t) \leq -\beta\alpha_1 F(t), \qquad \forall t \geq t_0. \tag{16}$$

A simple integration of (16) leads to

$$F(t) \leq F(t_0)e^{\beta\alpha_1 t_0}e^{-\beta\alpha_1 t}, \qquad \forall t \geq t_0. \tag{17}$$

Again by the (9), (17) yields

$$\mathcal{E}(t) \leq \alpha_2 F(t_0)e^{\beta\alpha_1 t_0}e^{-\beta\alpha_1 t}, \qquad \forall t \geq t_0. \tag{18}$$

This completes the proof.

Remark By repeating the same procedure, the same result holds for

$$\begin{gathered} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau \\ +a(x)u_t + b|u|^\gamma u = 0,\ b \neq 0 \text{ in } \Omega \times (0,\infty) \end{gathered}$$