Available online at www.sciencedirect.com
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J. Math. Anal. Appl. 320 (2006) 902–915
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation

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Received 17 December 2004
Available online 1 September 2005
Submitted by J. Lavery

Abstract

In this paper, we consider the nonlinear viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t|u_t|^{m-2} = u|u|^{p-2}$$

with initial conditions and Dirichlet boundary conditions. For nonincreasing positive functions g and for $p > m$, we prove that there are solutions with positive initial energy that blow up in finite time.

Keywords: Blow-up; Finite time; Hyperbolic; Nonlinear damping; Positive initial energy; Viscoelastic

1. Introduction

In this paper, we are concerned with the initial-boundary-value problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t|u_t|^{m-2} = u|u|^{p-2}, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \geqslant 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.07.022

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$ with a smooth boundary $\partial\Omega$, $p > 2$, $m \geqslant 1$, and g is a positive function. In the absence of the viscoelastic term (that is, if $g = 0$), the equation in (1.1) reduces to the nonlinearly damped wave equation

$$u_{tt} - \Delta u + u_t|u_t|^{m-2} = u|u|^{p-2}.$$

This equation has been extensively studied by many mathematicians. It is well known that in the further absence of the damping mechanism $u_t|u_t|^{m-2}$, the source term $u|u|^{p-2}$ causes finite-time blow-up of solutions with negative initial energy (see [1,9]). In contrast, in the absence of the source term, the damping term assures global existence for arbitrary initial data (see [8,10]). The interaction between the damping and source terms was first considered by Levine [11,12] for linear damping ($m = 2$). Levine showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [7] extended Levine's result to nonlinear damping ($m > 2$). In their work, the authors introduced a new method and determined relations between m and p for which there is global existence and other relations between m and p for which there is finite-time blow-up. Specifically, they showed that solutions with negative energy continue to exist globally if $m \geqslant p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative. Messaoudi [15] extended the blow-up result of [7] to solutions with only negative initial energy. For related results, we refer the reader to Levine and Serrin [13], Levine and Ro Park [14], Vitillaro [19], Yang [20] and Messaoudi and Said-Houari [18].

In the presence of the viscoelastic term ($g \neq 0$), Cavalcanti et al. [4] studied (1.1) for $m = 2$ and a localized damping mechanism $a(x)u_t$ ($a(x)$ null on a part of the domain). They obtained an exponential rate of decay by assuming that the kernel g is of exponential decay. This work was later improved by Cavalcanti et al. [6] and Berrimi and Messaoudi [2] using different methods. In related work, Cavalcanti et al. [3] studied solutions of

$$|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau - \gamma \Delta u_t = 0, \quad x \in \Omega,\ t > 0,$$

for $\rho > 0$ and proved a global existence result for $\gamma \geqslant 0$ and an exponential decay result for $\gamma > 0$. This latter result was extended by Messaoudi and Tatar [16] to a situation where a source term is competing with the damping induced by $-\gamma\Delta u_t$ and the integral term. Also, Cavalcanti et al. [5] established an existence result and a decay result for viscoelastic problems with nonlinear boundary damping.

Concerning nonexistence, Messaoudi [17] showed that Todorova and Georgiev's results can be extended to (1.1) using the technique of [7] with a modification in the energy functional due to the different nature of the problems.

In this article, we improve our earlier result by adopting and modifying the method of [19]. In particular, we will show that there are solutions of (1.1) with positive initial energy that blow up in finite time.

We first state a local existence theorem that can be established by combining arguments of [4,7].

Theorem 1.1. *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Let* $m > 1$, $p > 2$ *be such that*

$$\max\{m, p\} \leqslant \frac{2(n-1)}{n-2}, \quad n \geqslant 3. \tag{1.2}$$

Let g *be a* C^1 *function satisfying*

$$1 - \int_0^\infty g(s)\,ds = l > 0. \tag{1.3}$$

Then problem (1.1) *has a unique local solution*

$$u \in C\big([0, T_m); H_0^1(\Omega)\big), \quad u_t \in C\big([0, T_m); L^2(\Omega)\big) \cap L^m\big(\Omega \times (0, T_m)\big), \tag{1.4}$$

for some $T_m > 0$.

Remark 1.1. Condition (1.2) is needed to establish the local existence result (see [4,7]). In fact under this condition, the nonlinearity in the source is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$. Condition (1.3) is necessary to guarantee the hyperbolicity and well-posedness of system (1.1).

Next we state our main result. For this purpose, we assume that g satisfies, in addition to (1.3), the inequalities

$$\begin{aligned} &g(s) \geqslant 0, \qquad g'(s) \leqslant 0, \\ &\int_0^\infty g(s)\,ds < \frac{(p/2) - 1}{(p/2) - 1 + (1/2p)}. \end{aligned} \tag{1.5}$$

Theorem 1.2. *Let* m *and* p *be such that* $m > 1$, $p > \max\{2, m\}$ *and* (1.2) *holds. Assume further that* g *satisfies* (1.3), (1.5). *Then any solution of* (1.1) *with initial data satisfying* (2.7) *below blows up in finite time.*

2. Proof of the blow-up result

In this section we prove our main result (Theorem 1.2). For this purpose we let B be the best constant of the Sobolev embedding $[H^1] \hookrightarrow [L^p]$ and $B_1 = B/l^{1/2}$. We set

$$\alpha = B_1^{-p/(p-2)}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)\alpha^2. \tag{2.1}$$

We also define

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_p^p, \tag{2.2}$$

where

$$(g \circ v)(t) = \int_0^t g(t-\tau)\left\|v(t) - v(\tau)\right\|_2^2 \, d\tau.$$

Lemma 2.1. *Assume that* (1.2), (1.3) *and* (1.5) *hold. Let u be a solution of* (1.1). *Then $E(t)$is nonincreasing, that is,*

$$E'(t) \leqslant 0. \tag{2.3}$$

Proof. By multiplying Eq. (1.1) by u_t and integrating over Ω we obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\int_\Omega |u_t|^2 \, dx - \frac{1}{p}\int_\Omega |u|^p \, dx\right\} - \int_0^t g(t-\tau)\int_\Omega \nabla u_t(t).\nabla u(\tau)\, dx\, d\tau = -\int_\Omega |u_t|^m \, dx, \tag{2.4}$$

for any regular solution. This result remains valid for weak solutions by a simple density argument. For the last term on the left side of (2.4) we have

$$\begin{aligned}
&\int_0^t g(t-\tau)\int_\Omega \nabla u_t(t).\nabla u(\tau)\, dx\, d\tau \\
&\quad = \int_0^t g(t-\tau)\int_\Omega \nabla u_t(t).\left[\nabla u(\tau) - \nabla u(t)\right] dx\, d\tau \\
&\qquad + \int_0^t g(t-\tau)\int_\Omega \nabla u_t(t).\nabla u(t)\, dx\, d\tau \\
&\quad = -\frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\int_\Omega \left|\nabla u(\tau) - \nabla u(t)\right|^2 dx\, d\tau \\
&\qquad + \int_0^t g(\tau)\left(\frac{d}{dt}\frac{1}{2}\int_\Omega \left|\nabla u(t)\right|^2 dx\right) d\tau \\
&\quad = -\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-\tau)\int_\Omega \left|\nabla u(\tau) - \nabla u(t)\right|^2 dx\, d\tau\right] \\
&\qquad + \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(\tau)\int_\Omega \left|\nabla u(t)\right|^2 dx\, d\tau\right]
\end{aligned}$$

$$+\frac{1}{2}\int_0^t g'(t-\tau)\int_\Omega \left|\nabla u(\tau)-\nabla u(t)\right|^2 dx\,d\tau-\frac{1}{2}g(t)\int_\Omega \left|\nabla u(t)\right|^2 dx\,d\tau. \quad (2.5)$$

Inserting (2.5) into (2.4), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega |\nabla u|^2\,dx+\frac{1}{2}\int_\Omega |u_t|^2\,dx-\frac{1}{p}\int_\Omega |u|^p\,dx\right\}\\
&\quad+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-\tau)\int_\Omega \left|\nabla u(\tau)-\nabla u(t)\right|^2 dx\,d\tau\right]\\
&\quad-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(\tau)\left\|\nabla u(t)\right\|^2 d\tau\right]\\
&=-\int_\Omega |u_t|^m\,dx+\frac{1}{2}\int_0^t g'(t-\tau)\int_\Omega \left|\nabla u(\tau)-\nabla u(t)\right|^2 dx\,d\tau\\
&\quad-\frac{1}{2}g(t)\left\|\nabla u(t)\right\|_2^2\leqslant 0. \qquad (2.6)
\end{aligned}$$

This completes the proof. □

Lemma 2.2. *Assume that* (1.2), (1.3) *and* (1.5) *hold. Let u be a solution of* (1.1) *with initial data satisfying*

$$E(0)<E_1,\qquad \|\nabla u_0\|_2>B_1^{-p/(p-2)}. \quad (2.7)$$

Then there exists a constant $\beta>B_1^{-p/(p-2)}$ such that

$$\left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right]^{1/2}\geqslant\beta,\quad \forall t\in[0,T), \quad (2.8)$$

and

$$\|u\|_p\geqslant B_1\beta,\quad \forall t\in[0,T). \quad (2.9)$$

Proof. We first note that, by (2.2), we have

$$\begin{aligned}
E(t)&\geqslant\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{p}\|u\|_p^p\\
&\geqslant\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{p}B_1^p l^p\|\nabla u\|_2^p
\end{aligned}$$

$$\geqslant \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)$$
$$-\frac{B_1^p}{p}\left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right]^{p/2}$$
$$=\frac{1}{2}\zeta^2-\frac{B_1^p}{p}\zeta^p=h(\zeta), \tag{2.10}$$

where

$$\zeta=\left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right]^{1/2}.$$

It is easy to verify that h is increasing for $0<\zeta<\alpha$, decreasing for $\zeta>\alpha$, $h(\zeta)\to-\infty$ as $\zeta\to+\infty$, and

$$h(\alpha)=\left(\frac{1}{2}-\frac{1}{p}\right)B_1^{-2p/(p-2)}=E_1,$$

where α is given in (2.1). Therefore, since $E(0)<E_1$, there exists $\beta>\alpha$ such that $h(\beta)=E(0)$. If we set $\alpha_0=\|\nabla u_0\|_2$ then, by (2.10), we have

$$h(\alpha_0)\leqslant E(0)=h(\beta).$$

Therefore, $\alpha_0>\beta$.

To establish (2.8), we suppose by contradiction that

$$\left[\left(1-\int_0^{t_0} g(s)\,ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t_0)\right]^{1/2}<\beta,$$

for some $t_0>0$. By the continuity of

$$\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t),$$

we can choose t_0 such that

$$\left[\left(1-\int_0^{t_0} g(s)\,ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t_0)\right]^{1/2}>\alpha.$$

Again, the use of (2.10) leads to

$$E(t_0)\geqslant h\left(\left[\left(1-\int_0^{t_0} g(s)\,ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t_0)\right]^{1/2}\right)>h(\beta)=E(0).$$

This is impossible since $E(t)\leqslant E(0)$, for all $t\in[0,T)$. Hence (2.8) is established.

To prove (2.9), we exploit (2.2). We have

$$\frac{1}{2}\left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right]\leqslant E(0)+\frac{1}{p}\|u\|_p^p.$$

Consequently, we obtain

$$\begin{aligned}\frac{1}{p}\|u\|_p^p &\geqslant \frac{1}{2}\left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right]-E(0)\\ &\geqslant \frac{1}{2}\beta^2-E(0)\\ &\geqslant \frac{1}{2}\beta^2-h(\beta)=\frac{B_1^p}{p}\beta^p. \end{aligned}\tag{2.11}$$

The proof is complete. □

Lemma 2.3. *Suppose that* (1.2) *holds. Then there exists a positive constant* $C>1$ *such that*

$$\|u\|_p^s\leqslant C\big(\|\nabla u\|_2^2+\|u\|_p^p\big)\tag{2.12}$$

for any $u\in H_0^1(\Omega)$ and $2\leqslant s\leqslant p$.

Proof. If $\|u\|_p\leqslant 1$ then $\|u\|_p^s\leqslant\|u\|_p^2\leqslant C\|\nabla u\|_2^2$ by Sobolev embedding.

If $\|u\|_p>1$ then $\|u\|_p^s\leqslant\|u\|_p^p$. Therefore, (2.12) follows. This completes the proof. □

We set

$$H(t)=E_1-E(t)\tag{2.13}$$

and use, throughout this paper, C to denote a generic positive constant depending on p and l only. As a result of (2.2), (2.12), and (2.13), we have

Lemma 2.4. *Let u be solution of* (1.1). *Assume that* (1.2) *holds. Then we have*

$$\|u\|_p^s\leqslant C\big(-H(t)-\|u_t\|_2^2-(g\circ\nabla u)(t)+\|u\|_p^p\big),\quad \forall t\in[0,T),\tag{2.14}$$

for any $2\leqslant s\leqslant p$.

Proof. Using (1.3) and (2.2), we note that

$$\begin{aligned}\frac{1}{2}(1-l)\|\nabla u\|_2^2 &\leqslant \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2\\ &\leqslant E(t)-\frac{1}{2}\|u_t\|_2^2-\frac{1}{2}(g\circ\nabla u)(t)+\frac{1}{p}\|u\|_p^p\\ &\leqslant E_1-H(t)-\frac{1}{2}\|u_t\|_2^2-\frac{1}{2}(g\circ\nabla u)(t)+\frac{1}{p}\|u\|_p^p. \end{aligned}\tag{2.15}$$

Exploiting (2.1) and (2.9), simple calculations yield

$$E_1 \leqslant \frac{p-2}{2p}\|u\|_p^p. \tag{2.16}$$

Finally, a combination of (2.15) and (2.16) gives the desired result. □

Proof of Theorem 1.2. Using (2.2), (2.3) and (2.13), we obtain

$$\begin{aligned} 0 &< H(0) \leqslant H(t) \\ &\leqslant E_1 - \frac{1}{2}\left[\|u_t\|_2^2 + \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\right] + \frac{1}{p}\|u\|_p^p \end{aligned}$$

and, from (2.8), we obtain

$$\begin{aligned} &E_1 - \frac{1}{2}\left[\|u_t\|_2^2 + \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\right] \\ &\quad < E_1 - \frac{1}{2}\beta^2 = -\frac{1}{p}\beta^2 < 0, \quad \forall t \geqslant 0. \end{aligned} \tag{2.17}$$

Hence,

$$0 < H(0) \leqslant H(t) \leqslant \frac{1}{p}\|u\|_p^p, \quad \forall t \geqslant 0. \tag{2.18}$$

We define

$$L(t) := H^{1-\sigma}(t) + \varepsilon \int_\Omega u u_t\,dx, \tag{2.19}$$

for small ε to be chosen later and for

$$0 < \sigma \leqslant \min\left\{\frac{(p-2)}{2p}, \frac{(p-m)}{p(m-1)}\right\}. \tag{2.20}$$

Taking a derivative of (2.19) and using Eq. (1.1), we obtain

$$\begin{aligned} L'(t) &= (1-\sigma)H^{-\sigma}(t)\left\{\|u_t\|_m^m - \frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2\right\} \\ &\quad + \varepsilon \int_\Omega \left[u_t^2 - |\nabla u|^2\right] dx + \varepsilon \int_0^t g(t-\tau) \int_\Omega \nabla u(t).\nabla u(\tau)\,dx\,d\tau \\ &\quad + \varepsilon \int_\Omega |u|^p\,dx - \varepsilon \int_\Omega |u_t|^{m-2} u_t u\,dx \\ &\geqslant (1-\sigma)H^{-\sigma}(t)\|u_t\|_m^m + \varepsilon \int_\Omega \left[u_t^2 - |\nabla u|^2\right] dx \end{aligned}$$

$$+\varepsilon\int_\Omega |u|^p\,dx-\varepsilon\int_\Omega |u_t|^{m-2}u_t u\,dx+\varepsilon\int_0^t g(t-\tau)\|\nabla u(t)\|_2^2\,d\tau$$
$$+\varepsilon\int_0^t g(t-\tau)\int_\Omega \nabla u(t).\big[\nabla u(\tau)-\nabla u(t)\big]\,dx\,d\tau. \tag{2.21}$$

Using the Schwarz inequality, (2.21) takes on the form

$$L'(t)\geqslant(1-\sigma)H^{-\sigma}(t)\|u_t\|_m^m+\varepsilon\int_\Omega\big[u_t^2-|\nabla u|^2\big]\,dx$$
$$+\varepsilon\int_\Omega |u|^p\,dx-\varepsilon\int_\Omega |u_t|^{m-2}u_t u\,dx$$
$$-\varepsilon\int_0^t g(t-\tau)\big\|\nabla u(t)\big\|_2\big\|\nabla u(\tau)-\nabla u(t)\big\|_2\,d\tau$$
$$+\varepsilon\int_0^t g(t-\tau)\big\|\nabla u(t)\big\|_2^2\,d\tau. \tag{2.22}$$

We now exploit Young's inequality to estimate the fifth term on the right side of (2.22) and use (2.2) to substitute for $\int_\Omega |u(x,t)|^p\,dx$. Hence, (2.22) becomes

$$L'(t)\geqslant(1-\sigma)H^{-\sigma}(t)\|u_t\|_m^m+\varepsilon\int_\Omega u_t^2\,dx-\varepsilon\left(1-\int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|_2^2$$
$$+\varepsilon\left(pH(t)+\frac{p}{2}(g\circ\nabla u)(t)+\frac{p}{2}\|u_t\|_2^2+\frac{p}{2}\left(1-\int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|_2^2\right)$$
$$-\varepsilon\int_\Omega |u_t|^{m-2}u_t u(x,t)\,dx-\varepsilon\tau(g\circ\nabla u)(t)-\frac{\varepsilon}{4\tau}\int_0^t g(s)\,ds\big\|\nabla u(t)\big\|_2^2$$
$$\geqslant(1-\sigma)H^{-\sigma}(t)\|u_t\|_m^m+\varepsilon\left(1+\frac{p}{2}\right)\int_\Omega u_t^2\,dx+\varepsilon pH(t)$$
$$+\varepsilon\left(\frac{p}{2}-\tau\right)(g\circ\nabla u)(t)-\varepsilon\int_\Omega |u_t|^{m-2}u_t u\,dx$$
$$+\varepsilon\left(\left(\frac{p}{2}-1\right)-\left(\frac{p}{2}-1+\frac{1}{4\tau}\right)\int_0^\infty g(s)\,ds\right)\big\|\nabla u(t)\big\|_2^2, \tag{2.23}$$

for some number τ with $0<\tau<p/2$. Recalling (1.5), the estimate (2.23) reduces to

$$L'(t) \geqslant (1-\sigma)H^{-\sigma}(t)\|u_t\|_m^m + \varepsilon\left(1+\frac{p}{2}\right)\int_\Omega u_t^2(x,t)\,dx$$
$$+\varepsilon p H(t) + \varepsilon a_1(g\circ\nabla u)(t) + \varepsilon a_2\big\|\nabla u(t)\big\|_2^2 - \varepsilon\int_\Omega |u_t|^{m-2}u_t u\,dx, \quad (2.24)$$

where

$$a_1 = \frac{p}{2} - \tau > 0, \qquad a_2 = \left(\frac{p}{2}-1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\tau}\right)\int_0^\infty g(s)\,ds > 0.$$

To estimate the last term of (2.24), we again use Young's inequality

$$XY \leqslant \frac{\delta^r}{r}X^r + \frac{\delta^{-q}}{q}Y^q, \quad X, Y, \geqslant 0, \quad \forall\delta > 0, \quad \frac{1}{r} + \frac{1}{q} = 1$$

with $r = m$ and $q = m/(m-1)$. So we have

$$\int_\Omega |u_t|^{m-1}|u|dx \leqslant \frac{\delta^m}{m}\|u\|_m^m + \frac{m-1}{m}\delta^{-m/(m-1)}\|u_t\|_m^m,$$

which yields, by substitution in (2.24),

$$L'(t) \geqslant \left[(1-\sigma)H^{-\sigma}(t) - \frac{m-1}{m}\varepsilon\delta^{-m/(m-1)}\right]\|u_t\|_m^m$$
$$+\varepsilon\left(1+\frac{p}{2}\right)\int_\Omega u_t^2(x,t)\,dx + \varepsilon a_1(g\circ\nabla u)(t)$$
$$+\varepsilon a_2\big\|\nabla u(t)\big\|_2^2 + \varepsilon p H(t) - \varepsilon\frac{\delta^m}{m}\|u\|_m^m, \quad \forall\delta > 0. \quad (2.25)$$

Of course (2.25) remains valid even if δ is time-dependant since the integral is taken over the x variable. Therefore, taking δ so that $\delta^{-m/(m-1)} = kH^{-\sigma}(t)$ for large k to be specified later and substituting in (2.25), we arrive at

$$L'(t) \geqslant \left[(1-\sigma) - \frac{m-1}{m}\varepsilon k\right]H^{-\sigma}(t)\|u_t\|_m^m + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)\,dx$$
$$+\varepsilon a_1(g\circ\nabla u)(t) + \varepsilon a_2\big\|\nabla u(t)\big\|_2^2$$
$$+\varepsilon\left[pH(t) - \frac{k^{1-m}}{m}H^{\sigma(m-1)}(t)\|u\|_m^m\right]. \quad (2.26)$$

Exploiting (2.18) and the inequality $\|u\|_m^m \leqslant C\|u\|_p^m$, we obtain

$$H^{\sigma(m-1)}(t)\|u\|_m^m \leqslant \left(\frac{1}{p}\right)^{\sigma(m-1)} C\|u\|_p^{m+\sigma p(m-1)}.$$

Hence, (2.26) yields

$$L'(t) \geqslant \left[(1-\sigma) - \frac{m-1}{m}\varepsilon k\right] H^{-\sigma}(t)\|u_t\|_m^m + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)\,dx + \varepsilon a_1(g\circ\nabla u)(t) + \varepsilon a_2\big\|\nabla u(t)\big\|_2^2 + \varepsilon\left[pH(t) - \frac{k^{1-m}}{m}\left(\frac{1}{p}\right)^{\sigma(m-1)} C\|u\|_p^{m+\sigma p(m-1)}\right]. \tag{2.27}$$

We now use (2.20) and Lemma 2.4 with $s = m + \sigma p(m-1) \leqslant p$ to deduce from (2.27) that

$$\begin{aligned} L'(t) &\geqslant \left[(1-\sigma) - \frac{m-1}{m}\varepsilon k\right] H^{-\sigma}(t)\|u_t\|_m^m \\ &\quad + \varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)\,dx + \varepsilon a_1(g\circ\nabla u)(t) + \varepsilon a_2\big\|\nabla u(t)\big\|_2^2 \\ &\quad + \varepsilon\big[pH(t) - C_1 k^{1-m}\big\{-H(t) - \|u_t\|_2^2 - (g\circ\nabla u)(t) + \|u\|_p^p\big\}\big] \\ &\geqslant \left[(1-\sigma) - \frac{m-1}{m}\varepsilon k\right] H^{-\sigma}(t)\|u_t\|_m^m \\ &\quad + \varepsilon\left(\frac{p}{2}+1+C_1k^{1-m}\right)\|u_t\|_2^2 + \varepsilon\big(a_1 + C_1k^{1-m}\big)(g\circ\nabla u)(t) \\ &\quad + \varepsilon a_2\big\|\nabla u(t)\big\|_2^2 + \varepsilon\big(p + C_1k^{1-m}\big)H(t) - \varepsilon C_1 k^{1-m}\|u\|_p^p, \end{aligned} \tag{2.28}$$

where $C_1 = (1/p)^{\sigma(m-1)}C/m$. Noting that

$$H(t) \geqslant \frac{1}{p}\|u\|_p^p - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t)$$

and writing $p = 2a_3 + (p-2a_3)$, where $a_3 < \min\{a_1, a_2, p/2\}$, the estimate (2.28) yields

$$\begin{aligned} L'(t) &\geqslant \left[(1-\sigma) - \frac{m-1}{m}\varepsilon k\right] H^{-\sigma}(t)\|u_t\|_m^m \\ &\quad + \varepsilon\left(\frac{p}{2}+1+C_1k^{1-m} - a_3\right)\|u_t\|_2^2 + \varepsilon\big(a_1 + C_1k^{1-m} - a_3\big)(g\circ\nabla u)(t) \\ &\quad + \varepsilon(a_2 - a_3)\big\|\nabla u(t)\big\|_2^2 + \varepsilon\big(p - 2a_3 + C_1k^{1-m}\big)H(t) \\ &\quad + \varepsilon\left(\frac{2a_3}{p} - C_1k^{1-m}\right)\|u\|_p^p. \end{aligned} \tag{2.29}$$

At this point, we choose k large enough so that (2.29) becomes

$$L'(t) \geqslant \left[(1-\sigma) - \frac{m-1}{m}\varepsilon k\right] H^{-\sigma}(t)\|u_t\|_m^m + \varepsilon\gamma\big[H(t) + \|u_t\|_2^2 + \|u\|_p^p + (g\circ\nabla u)(t)\big], \tag{2.30}$$

where $\gamma > 0$ is the minimum of the coefficients of $H(t)$, $\|u_t\|_2^2$, $\|u\|_p^p$, and $(g\circ\nabla u)(t)$ in (2.29). Once k is fixed (hence γ also), we pick ε small enough so that

$$(1-\sigma) - \varepsilon k(m-1)/m \geqslant 0$$

and

$$L(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega u_0 u_1(x)\,dx > 0.$$

Therefore, (2.30) takes on the form

$$L'(t) \geqslant \varepsilon\gamma\big[H(t) + \|u_t\|_2^2 + \|u\|_p^p + (g \circ \nabla u)(t)\big]. \tag{2.31}$$

Consequently, we have

$$L(t) \geqslant L(0) > 0, \quad \forall t \geqslant 0.$$

We now estimate

$$\left|\int_\Omega u u_t\,dx\right| \leqslant \|u\|_2 \|u_t\|_2 \leqslant C\|u\|_p \|u_t\|_2,$$

which implies

$$\left|\int_\Omega u u_t\,dx\right|^{1/(1-\sigma)} \leqslant C\|u\|_p^{1/(1-\sigma)} \|u_t\|_2^{1/(1-\sigma)}.$$

Again, Young's inequality gives us

$$\left|\int_\Omega u u_t\,dx\right|^{1/(1-\sigma)} \leqslant C\big[\|u\|_p^{\mu/(1-\sigma)} + \|u_t\|_2^{\theta/(1-\sigma)}\big], \tag{2.32}$$

for $1/\mu + 1/\theta = 1$. To obtain $\mu/(1-\sigma) = 2/(1-2\sigma) \leqslant p$ by (2.20), we take $\theta = 2(1-\sigma)$. Therefore, (2.32) becomes

$$\left|\int_\Omega u u_t(x,t)\,dx\right|^{1/(1-\sigma)} \leqslant C\big[\|u\|_p^s + \|u_t\|_2^2\big],$$

where $s = 2/(1-2\sigma) \leqslant p$. Using Lemma 2.4, we obtain

$$\left|\int_\Omega u u_t\,dx\right|^{1/(1-\sigma)} \leqslant C\big[H(t) + \|u\|_p^p + \|u_t\|_2^2 + (g \circ \nabla u)(t)\big], \quad \forall t \geqslant 0. \tag{2.33}$$

Therefore, we have

$$\begin{aligned} L^{1/(1-\sigma)}(t) &= \left(H^{1-\sigma}(t) + \varepsilon \int_\Omega u u_t\,dx\right)^{1/(1-\sigma)} \\ &\leqslant 2^{1/(1-\sigma)}\left(H(t) + \left|\int_\Omega u u_t\,dx\right|^{1/(1-\sigma)}\right) \\ &\leqslant C\big[H(t) + \|u\|_p^p + \|u_t\|_2^2 + (g \circ \nabla u)(t)\big], \quad \forall t \geqslant 0. \end{aligned} \tag{2.34}$$

Combining (2.31) and (2.34), we arrive at

$$L'(t) \geqslant \Gamma L^{1/(1-\sigma)}(t), \quad \forall t \geqslant 0, \tag{2.35}$$

where Γ is a positive constant depending only on $\varepsilon\gamma$ and C. A simple integration of (2.35) over $(0, t)$ then yields

$$L^{\sigma/(1-\sigma)}(t) \geqslant \frac{1}{L^{-\sigma/(1-\sigma)}(0) - \Gamma t\sigma/(1-\sigma)}. \tag{2.36}$$

Therefore, (2.36) shows that $L(t)$ blows up in time

$$T^* \leqslant \frac{1-\sigma}{\Gamma\sigma[L(0)]^{\sigma/(1-\sigma)}}. \tag{2.37}$$

This completes the proof. □

Remark 2.1. By following the steps of the proof of Theorem 2.5 closely, one can easily see that the blow-up result holds even for $m = 1$ (damping caused only by viscosity). A small modification is needed in the proof.

Remark 2.2. The third inequality in (1.5) shows that there is a strong relation between the nonlinearity in the source and the damping caused by the viscosity. More precisely, the larger p is, the closer $\int_0^\infty g(s)\,ds$ can be to 1.

Remark 2.3. The estimate (2.37) shows that the larger $L(0)$ is, the quicker the blow-up takes place.

Acknowledgment

The author expresses his sincere thanks to KFUPM for its support. This work has been funded by KFUPM under Project # MS/VISCO ELASTIC/270.

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